ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

SMA

Sheaf Cohomology

Rafael GUGLIELMETTI

Semester project
Supervised by Prof. Eva BAYER FLUCKIGER
Assistant: Evis IERONYMOU

April 17, 2012

Contents

Abstract

The goal of this project is to introduce basic notions of Weil divisor, Cartier divisors Picard group, sheaf cohomology and to study links between these concepts.

Introduction

This project is divided in two parts. In the first one, we define sheaves of modules (as the sheaves extend the concept of abelian groups, sheaves of modules extend the notion of modules) and present some of their basic properties. Then, we speak about sheaves of modules on schemes and about coherent and quasi-coherent sheaves. The goal of the end of the first part is to present the notions of Weil and Cartier divisors and to show the link between them. Finally, we define the Picard group of a scheme and present the link with divisors.
The second chapter is about cohomology. First, we present the standard definitions of homological algebra which allow us to give the first definition of sheaf cohomology. Then, we present another version of cohomology: the Čech cohomology. This cohomology theory is more suitable for explicit calculations and coincides with the other one in many cases. Throughout this chapter, we illustrate these notions with various examples. Finally, we show how the Picard group can be seen as a cohomoloy group.

Chapter 1

Sheaves of modules and divisors

In this section, the term "sheaf" will always mean a sheaf of abelian groups and, unless specified otherwise, the underlying topological space of the sheaf is denoted X. I will use many of results etablished in [Gug10]. Most of these results can be easily updated to deal with $\mathcal{O}_X$-modules instead of sheaves and modules instead of abelian groups and I will not mention these modifications explicitely (except for few cases, as the sheafification process).

1.1 Sheaves of modules

1.1.1 Definitions and first properties

Definition 1.1.1 (Presheaf of modules)
Let $(X, \mathcal{O}_X)$ a ringed space. A presheaf of $\mathcal{O}_X$-modules *is a presheaf of abelian groups $\mathscr{F}$ such that $\mathscr{F}(U)$ is a $\mathcal{O}_X(U)$-module for all open set U of X and such that the restriction maps of $\mathscr{F}$ are compatible with the module structure, that is:*

$$(a \cdot s)\big|_V = a\big|_V \cdot s\big|_V, \quad \forall V \subset U \text{ open}, \forall a \in \mathcal{O}_X(U), \forall s \in \mathscr{F}(U).$$

Definition 1.1.2 (Sheaf of modules)
Let $(X, \mathcal{O}_X)$ a ringed space. A sheaf of $\mathcal{O}_X$-modules *$\mathscr{F}$, or an* $\mathcal{O}_X$-module*, is a presheaf of $\mathcal{O}_X$-modules such that $\mathscr{F}$ is a sheaf.*

Examples 1.1.3 (i) Each sheaf of abelian groups $\mathscr{F}$ is an $\mathbb{Z}$-module, where $\mathbb{Z}$ is viewed as the constant sheaf.

(ii) On a ringed space $\mathcal{O}_X$ is an $\mathcal{O}_X$-module (with the ring multiplication as action).

Proposition 1.1.4
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module and $x \in X$. Then $\mathscr{F}_x$ is an $\mathcal{O}_{X,x}$-module.

Proof. Let $[V, s] \in \mathscr{F}_x$ and $[W, a] \in \mathcal{O}_{X,x}$. We set

$$[W, a] \cdot [V, s] = \big[W \cap V, a\big|_{W\cap V} \cdot s\big|_{W\cap V}\big].$$

It is easy to see that this definition does not depend on the choice of the representatives (W, a) and (V, s). Since each $\mathscr{F}(U)$ is an $\mathcal{O}_X(U)$-module, this action makes $\mathscr{F}_x$ an $\mathcal{O}_{X,x}$-module. $\square$

Remark 1.1.5 (Stalk commutes with the action)
Let $a \in \mathcal{O}_X(U)$, $s \in \mathscr{F}(U)$ and $x \in X$. The way we define the action of $\mathcal{O}_{X,x}$ on $\mathscr{F}_x$ implies directly that $(a \cdot s)_x = a_x \cdot s_x$.

I recall here the process of sheafification which turns a presheaf into a sheaf.

Proposition 1.1.6
Let $\mathscr{F}$ be a presheaf of abelian groups on a topological space X. For each open set U of X, consider the set $\mathscr{F}^+(U)$ of functions $f : U \longrightarrow \coprod_{x\in U} \mathscr{F}_x$ which satisfy the two conditions:

($\star$) *For all $x \in U$, $f(x) \in \mathscr{F}_x$.*

($\star\star$) *For all $x \in U$, there exists an open neighbourhood V_x of x contained in U and an element $t \in \mathscr{F}(V_x)$ such that for all $y \in V_x$ the image t_y of t in $\mathscr{F}_y$ is equal to $f(y)$.*

Then $\mathscr{F}^+$ is a sheaf. Furthermore, if we define $\theta : \mathscr{F} \longrightarrow \mathscr{F}^+$ as

$$\begin{aligned}\theta_U : \mathscr{F}(U) &\longrightarrow \mathscr{F}^+(U)\\ g &\longmapsto \theta_U(g) : U \longrightarrow \textstyle\coprod_{y\in U} \mathscr{F}_y\\ &\qquad\qquad\quad x \longmapsto g_x,\end{aligned}$$

then θ is a morphism of presheaves and $(\mathscr{F}^+, \theta)$ satisfy the following universal property: for every sheaf $\mathscr{G}$ and every morphism of presheaves $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$, there exists a unique morphism β such that the following diagram commutes.

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\alpha"] \arrow[d, "\theta"'] & \mathscr{G} \\ \mathscr{F}^+ \arrow[ur, dashed, "\beta"'] & \end{tikzcd}$$

Proof. See 2.1.26 of [Gug10]. □

Definition 1.1.7 (Morphism of $\mathcal{O}_X$-modules)
Let $\mathscr{F}$ and $\mathscr{G}$ denote two $\mathcal{O}_X$-modules. A natural transformation $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules *if each component $\alpha_U : \mathscr{F}(U) \longrightarrow \mathscr{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. We denote by* $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ *the set of morphisms of $\mathcal{O}_X$-modules from $\mathscr{F}$ to $\mathscr{G}$.*

Remark 1.1.8
It is easy to see that the set $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ defined above endowed with the addition is an abelian group.

Notation 1.1.9
We denote by $\mathscr{M}od(\mathcal{O}_X)$ the category of $\mathcal{O}_X$-modules.

Definition 1.1.10 (Kernel, image and cokernel)
The kernel, *the* image *and the* cokernel, *which are denoted respectively* ker, im *and* coker, *of a morphism of $\mathcal{O}_X$-modules are defined in the same way that for a morphism of sheaves.*

Proposition 1.1.11
Let $\mathscr{F}$ and $\mathscr{G}$ denote two $\mathcal{O}_X$-modules and let $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. Then the sheaf $\ker\alpha$ *is an $\mathcal{O}_X$-module. Furthermore, this is the kernel of α in $\mathscr{M}od(\mathcal{O}_X)$.*

Proof. Let U be an open set of X. Since α_U is a morphism of $\mathscr{O}_X(U)$-modules, then $(\ker\alpha)(U) = \ker\alpha_U$ is an $\mathscr{O}_X(U)$-module. And since $(\ker\alpha)(U) \subset \mathscr{F}(U)$, then the restriction maps on $\ker\alpha$ are compatible with the module structure. It is clear that the inclusion map $i : \ker\alpha \longrightarrow \mathscr{F}$ is a morphism of $\mathscr{O}_X$-module. Let $\mathscr{F}'$ be an $\mathscr{O}_X$-module and $\alpha' : \mathscr{F}' \longrightarrow \mathscr{F}$ a morphism such that $\alpha\alpha' = 0$:

$$\begin{tikzcd}
\mathscr{F}' \arrow[d, dashed, "\beta"'] \arrow[dr, "\alpha'"] & & \\
\ker\alpha \arrow[r, hook, "i"'] & \mathscr{F} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "0"'] & \mathscr{G}
\end{tikzcd}$$

Then the morphism $\beta : \mathscr{F}' \longrightarrow \ker\alpha$ defined by $\beta_U = \alpha'_U$ is a morphism of $\mathscr{O}_X$-module which satisfy $i\,\beta = \alpha'$ and is uniquely so. Hence, $\ker\alpha$ is the kernel of α. □

Proposition 1.1.12
Let $\mathscr{F}$ be a presheaf of $\mathscr{O}_X$-modules. Then, the sheafification $\mathscr{F}^+$ of $\mathscr{F}$ is a $\mathscr{O}_X$-module and θ is a morphism of presheaves of $\mathscr{O}_X$-modules. Furthermore, in the universal property, if $\mathscr{G}$ is an $\mathscr{O}_X$-module and α is a morphism of presheaves of $\mathscr{O}_X$-modules, then the morphism of sheaves β is a morphism of $\mathscr{O}_X$-modules. In particular, the sheafification of a morphism of presheaves of $\mathscr{O}_X$-modules is a morphism of $\mathscr{O}_X$-modules.

Proof. Let U be an open set of X. We define the map

$$\begin{aligned}
\cdot : \mathscr{O}_X(U) \times \mathscr{F}^+(U) &\longrightarrow \mathscr{F}^+(U) \\
(s, f) &\longmapsto s \cdot f : U \longrightarrow \textstyle\bigcup_{y\in U} \mathscr{F}_y \\
&\qquad\qquad x \longmapsto s_x \cdot f(x),
\end{aligned}$$

where we use the action defined in Proposition 1.1.4. The condition $(*)$ is immediately verified. For the second one, we know that there exists for each $x \in U$ a neighbourhood V_x contained in U and an element $t \in \mathscr{F}(V_x)$ such that $f(y) = t_y$ for each $y \in V_x$. Then we use the element $s\big|_{V_x} \cdot t$ and the Remark 1.1.5. Now, it is easy to see that this map turn $\mathscr{F}^+(U)$ into a $\mathscr{O}_X(U)$-module. Furthermore, if $V \subset U$, $x \in V$, $s \in \mathscr{O}_X(U)$ and $f \in \mathscr{F}^+(U)$, then

$$\big(s\big|_V \cdot f\big|_V\big)(x) = \big(s\big|_V\big)_x \cdot f(x) = s_x \cdot f(x) = (s \cdot f)\big|_V(x).$$

Hence, the action is compatible with the restriction maps and $\mathscr{F}^+$ is an $\mathscr{O}_X$-module, as required. A direct calculation shows that θ_U is a morphism of $\mathscr{O}_X(U)$-modules. Reasoning like in the proof of Proposition 1.1.6 show that the induced morphism β is a morphism of $\mathscr{O}_X$-modules. □

Proposition 1.1.13
In $\mathscr{M}od(\mathscr{O}_X)$, monomorphisms are exactly injective morphisms (i.e. morphisms α such that $\ker\alpha$ *is trivial).*

Proposition 1.1.14
Let $\mathscr{F}$ and $\mathscr{G}$ denote two $\mathscr{O}_X$-modules and let $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules. Then the sheaf $\operatorname{im}\alpha$ *is an $\mathscr{O}_X$-module and is the image of α in $\mathscr{M}od(\mathscr{O}_X)$.*

Proof. In this proof, we made an exception and use $\operatorname{im}\alpha$ for the presheaf image and $(\operatorname{im}\alpha)^+$ for the image of α. For every open set U of X, $\alpha\big(\mathscr{F}(U)\big)$ has a natural structure of $\mathcal{O}_X(U)$-module. Hence, $\operatorname{im}\alpha$ is a presheaf of $\mathcal{O}_X$-modules. The previous Proposition shows that $(\operatorname{im}\alpha)^+$ is an $\mathcal{O}_X$-module.
We have the following situation:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\alpha"] & \operatorname{im}\alpha \arrow[r, "i"] \arrow[d, "\theta"'] & G \\
& (\operatorname{im}\alpha)^+ \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Since i is a monomorphism, then so is β (Proposition 2.1.36 of [Gug10]). We want to show that $\big((\operatorname{im}\alpha)^+, \beta\big)$ satisfy the universal property of the image: for every $\mathcal{O}_X$-module $\mathscr{F}'$, for every morphism $\gamma : \mathscr{F} \longrightarrow \mathscr{F}'$ and every injective morphism $\delta : \mathscr{F}' \longrightarrow \mathscr{G}$ such that $\alpha = \delta\gamma$, there exists a unique morphism $\varepsilon : (\operatorname{im}\alpha)^+ \longrightarrow \mathscr{F}'$ making the following diagram commute:

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, bend left, "\alpha"] \arrow[dr, "\theta\alpha"] \arrow[ddr, bend right, "\gamma"'] & & \mathscr{G} \\
& (\operatorname{im}\alpha)^+ \arrow[ur, "\beta"'] \arrow[d, dashed, "\varepsilon"] & \\
& \mathscr{F}' \arrow[uur, bend right, "\delta"'] &
\end{tikzcd}$$

Suppose that γ and δ are as above. For all $t \in (\operatorname{im}\alpha)_U = \alpha_U\big(\mathscr{F}(U)\big)$, let $s \in \mathscr{F}(U)$ be a preimage of t under α_U and set $\eta_U(t) = \gamma_U(s)$. Since δ is injective, the definition of $\eta_U(t)$ does not depend on the choice of the preimage. This definition gives a morphism $\eta : \operatorname{im}\alpha \longrightarrow \mathscr{F}'$ of presheaves of $\mathcal{O}_X$-modules and the universal property gives the desired morphism ε. We have to check that the morphism ε makes the diagram commute. We have $\varepsilon\theta\alpha = \eta\alpha = \gamma$. For the second triangle, we get $\delta\varepsilon\theta = \delta\eta = i$. Since $\beta\theta = i$, the universal property implies $\delta\varepsilon = i$. If ε' is another morphism which makes the diagram commute, then

$$\delta\theta\varepsilon' = i = \delta\theta\varepsilon \Rightarrow \theta\varepsilon' = \theta\varepsilon \Rightarrow \varepsilon' = \varepsilon,$$

as required. □

Proposition 1.1.15
Let $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. Then the cokernel of α is an $\mathcal{O}_X$-module and the cokernel of α in $\mathscr{M}od(\mathcal{O}_X)$.

Proof. In this proof, we made an exception and use $\operatorname{coker}\alpha$ for the presheaf coker and $(\operatorname{coker}\alpha)^+$ for the cokernel of α. Using Proposition 1.1.12, it is easy to see that $(\operatorname{coker}\alpha)^+$ has a natural structure of $\mathcal{O}_X$-module. If $\beta : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of $\mathcal{O}_X$-modules such that $\beta\alpha = 0$, then the universal property of the cokernel of modules induces the components of a morphism γ and the universal property of the sheafification induces the required morphism δ. The situation is the following:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "0"'] & \mathscr{G} \arrow[r, "\pi"] \arrow[rrd, "\beta"'] & \operatorname{coker}\alpha \arrow[r, "\theta"] \arrow[rd, dashed, "\gamma"] & (\operatorname{coker}\alpha)^{+} \arrow[d, dashed, "\delta"] \\
 & & & \mathscr{G}'
\end{tikzcd}$$

$\square$

So far, we have seen that:

(i) For all $\mathscr{F}, \mathscr{G} \in \mathscr{M}od(\mathcal{O}_X)$, $\mathrm{Hom}(\mathscr{F}, \mathscr{G})$ as the structure of an abelian group and the composition law is linear.

(ii) Every morphism in $\mathscr{M}od(\mathcal{O}_X)$ has a kernel and a cokernel.

(iii) Finite direct sums exist.

And it is easy to check the following:

(i) Every monomorphism is the kernel of its cokernel.

(ii) Every epimorphism is the cokernel of its kernel.

(iii) Each morphism $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ can be factored into

$$\begin{tikzcd}
\mathscr{F} \arrow[rr, "\alpha"] \arrow[rd, two heads] & & \mathscr{G} \\
 & \mathscr{F}' \arrow[ru, hook] &
\end{tikzcd}$$

Therefore, we have the following result:

Proposition 1.1.16
The category $\mathscr{M}od(\mathcal{O}_X)$ is an abelian category.

Definition 1.1.17 ($\mathcal{O}_X$-submodule)
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. A subsheaf $\mathscr{F}'$ of $\mathscr{F}$ is a subsheaf of $\mathcal{O}_X$-modules, *or a* $\mathcal{O}_X$-submodule, *of $\mathscr{F}$ if the two following conditions are satisfied:*

(i) $\mathscr{F}'(U)$ is a submodule of $\mathscr{F}(U)$ for each open set U of X;

(ii) the restriction maps of $\mathscr{F}'$ are induced by those on $\mathscr{F}$.

Definition 1.1.18 (Quotient sheaf)
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module and $\mathscr{F}'$ an $\mathcal{O}_X$-submodule of $\mathscr{F}$. The quotient sheaf $\mathscr{F}/_{\mathscr{F}'}$ is defined as for sheaves: this is the sheaf associated to the presheaf which maps U to $\mathscr{F}(U)/_{\mathscr{F}'(U)}$.

The following proposition is an immediate consequence of the Proposition 1.1.12.

Proposition 1.1.19
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module and $\mathscr{F}'$ an $\mathcal{O}_X$-submodule of $\mathscr{F}$. Then the quotient sheaf $\mathscr{F}/_{\mathscr{F}'}$ is an $\mathcal{O}_X$-module.

Definition 1.1.20 (Exact sequence of $\mathcal{O}_X$-modules)
A sequence of $\mathcal{O}_X$-modules is exact *if it is exact as a sequence of sheaves of abelian groups.*

Proposition 1.1.21
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module and U an open set of X. Then $\mathscr{F}\big|_U$ is an $\mathcal{O}_X\big|_U$-module.

Definition 1.1.22 (Sheaf hom)
Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$-modules. Then the presheaf $U \longmapsto \operatorname{Hom}_{\mathcal{O}_X|_U}\left(\mathscr{F}\big|_U, \mathscr{G}\big|_U\right)$ is denoted $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$. It is a sheaf and is called the sheaf hom.

Proposition 1.1.23
For two $\mathcal{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, the sheaf hom $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is an $\mathcal{O}_X$-module.

Definition 1.1.24 (Tensor product of $\mathcal{O}_X$-modules)
Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$ modules. We define the tensor product *of $\mathscr{F}$ and $\mathscr{G}$, denoted $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ or $\mathscr{F} \otimes \mathscr{G}$ if no confusion can arise, to be the sheaf associated to the presheaf $U \longmapsto \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$.*

Proposition 1.1.25
The tensor product of two $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module.

Example 1.1.26
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Then $\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathscr{F}$. Indeed, for every open set U we have $\mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U) \cong \mathscr{F}(U)$.

Proposition 1.1.27
Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathcal{O}_X$-modules. Then, for all $x \in X$, we have

$$\left(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}\right)_x \cong \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x,$$

as $\mathcal{O}_{X,x}$-modules.

Proof. Fix $x \in X$. We have the following situation

$$\begin{tikzcd}
\mathscr{F}_x \times \mathscr{G}_x \arrow[r, "\varphi"] \arrow[d, "\tau"'] & \varinjlim_{x\in U} \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U) \arrow[dl, dashed, bend left, "\tilde{\psi}"] \\
\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x \arrow[ur, dashed, bend left, "\tilde{\varphi}"] &
\end{tikzcd}$$

where:

- $\varphi\big([V,s],[W,t]\big) = \big[V \cap W, s\big|_{V\cap W} \otimes t\big|_{V\cap W}\big]$;
- $\tilde{\varphi}$ is induced by the universal property of the tensor product.
- $\tilde{\psi}\big((s \times t)_x\big) = s_x \otimes t_x$. To get this map, first define

  $$\psi_U : \mathscr{F}(U) \times \mathscr{G}(U) \longrightarrow \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x$$

  by $\psi_U(s,t) = s_x \otimes t_x$. Then use the universal property of tensor product to get $\tilde{\psi}_U$ and finally use the universal property of the direct limit to get the desired morphism $\tilde{\psi}$.

Since these maps are mutual inverse and stalks commutes with sheafification, we get the required result. ∎

Definition 1.1.28 (Free $\mathcal{O}_X$-module)
An $\mathcal{O}_X$-module is free *if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$.*

Definition 1.1.29 (Locally free $\mathcal{O}_X$-module)
An $\mathcal{O}_X$-module $\mathscr{F}$ is locally free *if there exists an open covering $\{U_i\}_{i\in I}$ of X such that each $\mathscr{F}\big|_{U_i}$ is free (as an $\mathcal{O}_X\big|_{U_i}$-module).*

Definition 1.1.30
In the above proposition, the rank of an open set U_i is the rank of $\mathscr{F}\big|_{U_i}$ as an $\mathcal{O}_X\big|_{U_i}$-module.

Proposition 1.1.31
Let $\mathscr{F}$ be an $\mathcal{O}_X$-module and U an open set such that $\mathscr{F}\big|_U$ is free of rank n (with $n \in \mathbb{N}_0 \cup \{\infty\}$). Then for all $x \in U$, $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank n.

Proof. We recall that for any sheaf $\mathscr{G}$, any open set U and any $x \in U$, we have $\big(\mathscr{G}\big|_U\big)_x \cong \mathscr{G}_x$. Furthermore, it is easy to see that taking stalks commute with direct sum. $\square$

Proposition 1.1.32
Let $\mathscr{F}$ be a locally free $\mathcal{O}_X$-module over a connected topological space X. Then the rank is the same everywhere.

Proof. Consider the function $f : X \longrightarrow \mathbb{N} \cup \{\infty\}$ which associate to each $x \in X$ the rank of $\mathscr{F}_x$ as an $\mathcal{O}_{X,x}$-module. The last Proposition implies that this function is locally constant. Since X is connected, f is constant. $\square$

Definition 1.1.33 (Rank of a locally free $\mathcal{O}_X$-module)
Let $\mathscr{F}$ be a locally free $\mathcal{O}_X$-module such that the rank of $\mathscr{F}$ on each open set of the open covering is the same. Then, the rank *of $\mathscr{F}$ is the rank of $\mathscr{F}$ on some open set. Remark that Proposition 1.1.31 implies that the rank does not depend on the choice of the covering.*

Definition 1.1.34 (Invertible sheaf)
A locally free $\mathcal{O}_X$-module of rank 1 *is called an* invertible sheaf *(or* invertible $\mathcal{O}_X$-module*).*

Remark 1.1.35
We will see later (cf. Section 1.2.3) the reason for the choice of the term "invertible".

Definition 1.1.36
A sheaf of ideals *on X is a sheaf of modules $\mathscr{G}$ which is a subsheaf of $\mathcal{O}_X$. That is, $\mathscr{G}(U)$ is an ideal of $\mathcal{O}_X(U)$.*

1.1.2 Direct and inverse images

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces[1]. If $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathscr{F}$ is an $f_*\mathcal{O}_X$-module by the evident action. On the other hand, we can put a structure of $\mathcal{O}_Y$-module on $f_*\mathscr{F}$:

$$\begin{aligned} \mathcal{O}_Y(V) \times \big(f_*\mathscr{F}\big)(V) &\longrightarrow \big(f_*\mathscr{F}\big)(V) \\ (a, s) &\longmapsto a \cdot s = f^\sharp_V(a) \cdot s. \end{aligned}$$

[1]We recall that a morphism of ringed spaces $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ where $f : X \longrightarrow Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is a morphism of sheaves.

If $W \subset V$, we have

$$(a \cdot s)\big|_W = f_V^\sharp(a)\big|_W \cdot s\big|_W = f_W^\sharp\big(a\big|_W\big) \cdot s\big|_W = a\big|_W \cdot s\big|_W.$$

Therefore, the action is compatible with the restriction maps.

Definition 1.1.37 (Direct image)
Let f and $\mathscr{F}$ as above. The $\mathcal{O}_Y$-module $f_*\mathscr{F}$ is called the direct image *of $\mathscr{F}$ by $(f, f^\sharp)$.*

Remark 1.1.38
If $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules, then $f_*\alpha : f_*\mathscr{F} \longrightarrow f_*\mathscr{G}$ is a morphism of $\mathcal{O}_Y$-modules. Hence, f_* is a functor from $\mathscr{M}od(\mathcal{O}_X)$ to $\mathscr{M}od(\mathcal{O}_Y)$

Proposition 1.1.39
Let $\mathscr{G}$ be an $\mathcal{O}_Y$-module and $f : X \longrightarrow Y$ a continous map. Then $f^{-1}\mathscr{G}$ is a $f^{-1}\mathcal{O}_Y$-module.

Proof. For any open set U of X, we define the following action:

$$\begin{aligned} \varinjlim_{f(U)\subset V} \mathcal{O}_Y(V) \times \varinjlim_{f(U)\subset V} \mathscr{G}(V) &\longrightarrow \varinjlim_{f(U)\subset V} \mathscr{G}(V) \\ \big([V,a],[W,t]\big) &\longmapsto \big[V \cap W, a\big|_{V\cap W} \cdot t\big|_{V\cap W}\big]. \end{aligned}$$

□

Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Since f^{-1} and f_* are adjoint functors, $f^\sharp$ leads to a morphism $f_\sharp : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ which defines a a structure of $f^{-1}\mathcal{O}_Y$-module on $\mathcal{O}_X$. Then we set

$$f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

which is an $\mathcal{O}_X$-module.

Definition 1.1.40
Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces and $\mathscr{G}$ an $\mathcal{O}_Y$-module. Then the $\mathcal{O}_X$-module $f^*\mathscr{G}$ defined above is called the inverse image *of $\mathscr{G}$ by f.*

Remark 1.1.41
Let U be an open subset of X and $i : U \hookrightarrow X$ the inclusion. For a sheaf $\mathscr{F}$ on X, we defined the restriction $\mathscr{F}\big|_U$ as $i^{-1}(U)$ and if $\mathscr{F}$ is an $\mathcal{O}_X$-module, then $\mathscr{F}\big|_U$ is an $\mathcal{O}_X\big|_U$-module. Now, we have the concept of inverse image, a map $i : \big(U, \mathcal{O}_X\big|_U\big) \longrightarrow (X, \mathcal{O}_X)$ and we can see that $i^*\mathscr{F} \cong i^{-1}\mathscr{F}$ (as $\mathcal{O}_X\big|_U$-modules). But this is not necessarily the case when i is not an open immersion. For example, consider $X = \operatorname{Spec}\mathbb{C}[t]$, $a \in \mathbb{C}$ and the point $P_a = \langle t-a\rangle \in \operatorname{Spec}\mathbb{C}[t]$. Let $Y = \operatorname{Spec}\mathbb{C}$ and $i : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X)$, where $i(0) = P_a$. Then $i^{-1}\mathcal{O}_X \cong \mathcal{O}_{X,P_a} \cong \mathbb{C}[t]_{\langle x-a\rangle}$. On the other hand, we have

$$i^*\mathcal{O}_X = i^{-1}\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_X \cong \mathbb{C}[t]_{\langle t-a\rangle} \otimes_{\mathbb{C}[t]_{\langle t-a\rangle}} \mathbb{C},$$

which is isomorphic to $\mathbb{C}$.

Proposition 1.1.42
Let f and $\mathscr{G}$ as in the definition. Then for all $x \in X$ we have

$$(f^*\mathscr{G})_x \cong \mathscr{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

Proof. Use Proposition 1.1.27 and Remark 2.1.54 of [Gug10]. □

Remark 1.1.43
If $\alpha : \mathscr{G} \longrightarrow \mathscr{G}'$ is a morphism of $\mathcal{O}_X$-modules, the functoriality of f^{-1} gives a morphism $f^{-1} : f^{-1}\mathscr{G} \longrightarrow f^{-1}\mathscr{G}'$ of $f^{-1}\mathcal{O}_Y$-modules. Then we get a morphism of $\mathcal{O}_X$-modules $\alpha^* : f^*\mathscr{G} \longrightarrow f^*\mathscr{G}'$. Hence, f^* is a functor from $\mathscr{M}od(\mathcal{O}_Y)$ to $\mathscr{M}od(\mathcal{O}_X)$.

1.1.2.1 Exacteness of direct and inverse image functors

Here, we discuss briefly the exacteness of the direct and inverse image functors.

Proposition 1.1.44
The inverse image functor is right-exact and the direct image functor is left-exact.

Proof. One can show that the functor f^* is left adjoint to the functor f_*. Hence, f^* is right-exact and f_* is left-exact. □

Remark 1.1.45
For the inverse image, one can check this directly: we know that f^{-1} is an exact functor. Furthermore, the functor $-\otimes M$ is right-exact, for any module M for which tensor product makes sense.

Proposition 1.1.46
In the case of closed/open immersions, we have the following stronger result:

(*i*) *Let U be an open subset of X and $i : \left(U, \mathcal{O}_X\big|_U\right) \longrightarrow (X, \mathcal{O}_X)$ be the inclusion morphism. Then the functor i^* is exact.*

(*ii*) *If $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a closed immersion, then f_* is exact.*

Proof. (i) Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Using last Proposition, we get $(i^*\mathscr{F})_x \cong \mathscr{F}_x$. The result follows from the fact that exacteness can be checked on the stalks.

(ii) We know already that f_* is left-exact. We consider a surjective morphism $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ of $\mathcal{O}_X$-modules and want to show that $f_*\alpha$ is surjective, which is equivalent to $(f_*\alpha)_y$ being surjective for all $y \in Y$. First, we remark that since $f(X)$ is closed in Y, then $(f_*\mathscr{G})_y = 0$ for all $y \notin f(X)$. Now, let $y = f(x)$. Since f is a closed immersion, the set $\left\{f^{-1}(V) : V \text{ open neighbourhood of } y\right\}$ is cofinal in the set of open neighbourhood of x. This implies that $\mathscr{F}_x \cong (f_*\mathscr{F})_y$ and $\mathscr{G}_x \cong (f_*\mathscr{G})_y$. Therefore, we have the following commutating diagram

$$\begin{tikzcd} \mathscr{F}_x \arrow[r, "\cong"] \arrow[d] & (f_*\mathscr{F})_y \arrow[d, "(f_*\alpha)_y"] \\ \mathscr{G}_x \arrow[r, "\cong"] & (f_*\mathscr{F})_y \end{tikzcd}$$

which implies that $(f_*\alpha)_y$ is surjective, as required.

□

1.1.3 Sheaves of modules on an affine scheme

Let R be a ring, $(\operatorname{Spec} R, \mathcal{O}_R)$ its spectrum and M be a R-module. We want to use M to define a sheaf of $\mathcal{O}_R$-modules $\tilde{M}$. To do this, we copy the construction of the structure sheaf $\mathcal{O}_R$ of $\operatorname{Spec} R$. Let U be an open set of $\operatorname{Spec} R$ and set $\tilde{M}(U)$ as the set of all functions $s : U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}}$ such that:

- $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.
- For all $\mathfrak{p} \in U$ there exists an open neighbourhood V of $\mathfrak{p}$ contained in U and $m \in M$, $r \in R$ such that for each $\mathfrak{q} \in V$, $r \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{r}$.

With the restriction as restriction maps, we get a sheaf. We define a structure of $\mathcal{O}_R(U)$-module on $\tilde{M}(U)$ as follows:

$$\begin{aligned} \mathcal{O}(U) \times \tilde{M}(U) &\longrightarrow \tilde{M}(U) \\ (f, s) &\longmapsto f \cdot s : \mathfrak{p} \longmapsto f(\mathfrak{p}) \cdot s(\mathfrak{p}). \end{aligned}$$

As restriction of functions, the restriction maps are compatible with the module structure.

Definition 1.1.47 (Sheaf associated to a module)
Let R be a ring, $(\operatorname{Spec} R, \mathcal{O})$ its spectrum and M be an R-module. The $\mathcal{O}$-module $\tilde{M}$ defined above is called the sheaf associated to M on $\operatorname{Spec} R$.

It is clear from the definitions that considering R as a module over itself then $\tilde{R}$ is just $\mathcal{O}_R$. Recall that for an element $f \in R$, we define $D(f) = \operatorname{Spec} R \setminus \mathcal{V}(\langle f \rangle)$.

The following two propositions give the main properties of the construction we made.

Proposition 1.1.48
Let R be a ring, $X = \operatorname{Spec} R$ and M a R-module. Then:

(i) For each $\mathfrak{p} \in X$, we have $\tilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$ (as $\mathcal{O}_{X,\mathfrak{p}}$-modules).

(ii) For every $f \in R$, we have $\tilde{M}\big(D(f)\big) \cong M_f$ (as R_f-modules).

(iii) In particular, $\tilde{M}(\operatorname{Spec} R) \cong M$.

Proof. See [Har77] II.5. □

Proposition 1.1.49
Let R be a ring and $X = \operatorname{Spec} R$. Let $R \longrightarrow S$ be a homomorphism of rings and the corresponding application $f : \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$. Then:

(i) The map $M \longmapsto \tilde{M}$ gives an exact, fully faithful functor from the category of R-modules to $\mathscr{M}od(\mathcal{O}_X)$.

(ii) For any family of R-modules $\{M_i\}$, we have $\big(\bigoplus_i M_i\big)^{\sim} \cong \bigoplus_i \tilde{M}_i$.

(iii) Let M, N be a pair of R-modules, then $(M \otimes_R N)^{\sim} \cong \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$.

(iv) If N is a S-module, then we have $f_*(\tilde{N}) \cong \big({}_R N\big)^{\sim}$.

(v) For any R-module M, we have $f^*(\tilde{M}) \cong \big(M \otimes_R S\big)^{\sim}$.

Proof. (i) First we describe the functor on morphisms. Let $\varphi : M \longrightarrow N$ be a morphism of R-modules. For every open set U of X, we set:

$$\begin{aligned}\tilde{\varphi}_U : \tilde{M}(U) &\longrightarrow \tilde{N}(U)\\ s &\longmapsto \tilde{\varphi}_U(s) : \mathfrak{p} \longmapsto \varphi_{\mathfrak{p}}\big(s(\mathfrak{p})\big),\end{aligned}$$

where $\varphi_{\mathfrak{p}} : M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}}$ is induced by φ.
The important fact is that, with the identification $\tilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$, the morphism between $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ induced by φ is $\varphi_{\mathfrak{p}}$. Thus, a sequence of $\mathcal{O}_X$-modules $\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N} \xrightarrow{\tilde{\psi}} \tilde{P}$ is exact at $\tilde{N}$ if and only if the sequence $M_{\mathfrak{p}} \xrightarrow{\varphi_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{\psi_{\mathfrak{p}}} P_{\mathfrak{p}}$ is exact at $N_{\mathfrak{p}}$ (Corollary 2.1.43 of [Gug10]). Since localization at a prime ideal is an exact functor, then so is $\sim$.
Let M and N be two R-modules. We have an application $\Phi_{M,N}$ which sends a morphism φ to $\tilde{\varphi}$. On the other hand, given a morphism $\alpha : \tilde{M} \longrightarrow \tilde{N}$, we can use the point (iii) of the previous proposition, to get a morphism $\Psi_{M,N} : M \longrightarrow N$. Then one can check that these applications are mutually inverses.

(ii) The universal property gives rise to a morphism $\alpha : \bigoplus_i \tilde{M}_i \longrightarrow \big(\bigoplus_i M_i\big)^{\sim}$. By looking at stalks and noting that $\big(\bigoplus_i M_i\big)_{\mathfrak{p}} \cong \bigoplus_i \big(M_i\big)_{\mathfrak{p}}$ (see Proposition A.1.1) we see that α is an isomorphism.

(iii) Using the fact that $(M \otimes_R N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$, we can construct a map

$$\begin{aligned}\alpha_U : \tilde{M}(U) \times \tilde{N}(U) &\longrightarrow (M \otimes_R N)^{\sim}(U)\\ (s,t) &\longmapsto \alpha_U(s,t) : \mathfrak{q} \longmapsto s(\mathfrak{q}) \otimes t(\mathfrak{q}),\end{aligned}$$

which gives a map $\tilde{\alpha}_U : \tilde{M}(U) \otimes_{\mathcal{O}_X(U)} \tilde{N}(U) \longrightarrow (M \otimes_R N)^{\sim}(U)$. Since the induced map $\tilde{\alpha}_{\mathfrak{q}}$ is an isomorphism for each $\mathfrak{q} \in X$ and since stalks are preserved by sheafification, we have the required isomorphism.

(iv) Follows from the definitions.

(v) For every $\mathfrak{q} \in \operatorname{Spec} S$, consider the following canonical isomorphism:

$$\tau_{\mathfrak{q}} : \big(f^*\tilde{M}\big) \longrightarrow \big(M \otimes_R S\big)_{\mathfrak{q}}$$

For every open set V of $\operatorname{Spec} S$, set

$$\mathscr{F}(V) = \big(f^{-1}\tilde{M}\big)(V) \otimes_{(f^{-1}\mathcal{O}_R)(V)} \mathcal{O}_S(V),$$

which means that $\mathscr{F}^+ = f^*\tilde{M}$. Then, we define

$$\begin{aligned}\alpha : \mathscr{F} &\longrightarrow \big(M \otimes_R S\big)^{\sim}\\ \alpha_V(s) : V &\longrightarrow \coprod_{\mathfrak{p}\in V} \big(M \otimes_R S\big)_{\mathfrak{q}}, \alpha_V(s) : \mathfrak{q} \longmapsto \tau_{\mathfrak{q}}(s_{\mathfrak{q}}).\end{aligned}$$

One can check that the maps α_V are well defined and that α is a morphism of modules. Furthermore, we have $\alpha_{\mathfrak{q}} = \tau_{\mathfrak{q}}$ which implies that α is an isomorphism. $\square$

1.1.4 Quasi-coherent and coherent sheaves

Definition 1.1.50 (Quasi-coherent sheaf, coherent sheaf)
Let $(X, \mathcal{O}_X)$ be a scheme. An $\mathcal{O}_X$-module $\mathscr{F}$ is quasi-coherent *if X can be covered by open affine subsets $U_i = \operatorname{Spec} R_i$, such that for each i there exists an R_i-module M_i with $\mathscr{F}\big|_{U_i} \cong \tilde{M}_i$ (as $\mathcal{O}_{\operatorname{Spec} R_i}$-modules). If in addition each M_i can be taken to be a finitely generated R_i-module, then we say that $\mathscr{F}$ is* coherent.

Example 1.1.51
Let $(X, \mathcal{O}_X)$ be a scheme. Then there exists an open covering $\{U_i = \operatorname{Spec} R_i\}$ of X such that $\mathcal{O}_X\big|_{U_i} \cong \tilde{R}_i$. Therefore, $\mathcal{O}_X$ is coherent.

Proposition 1.1.52
Let $\mathscr{F}$ be a quasi-coherent sheaf on an affine scheme $(X = \operatorname{Spec} R, \mathcal{O}_X)$. Then there exists $n \in \mathbb{N}$, $f_1, \ldots, f_n \in R$ and R_{f_i}-modules M_i, such that:

(i) X is covered by the $D(f_i)$;

(ii) $\mathscr{F}\big|_{D(f_i)} \cong \tilde{M}_i$.

Proof. We know that $\{D(f) : f \in R\}$ is a base for the topology on $\operatorname{Spec} R$. Furthermore, there exists an open covering $\{U_i\}_{i\in I}$ and a collection of R_i-modules M_i such that $U_i \cong \operatorname{Spec} R_i$ and $\mathscr{F}\big|_{U_i} \cong \tilde{M}_i$. Fix an U_i and $f \in R$ such that $D(f) \subset U_i$. The inclusion of $D(f)$ in U_i induce a morphism of rings from $\mathcal{O}_X\big(D(f)\big) = R_f$ to $\mathcal{O}_X(U_i) \cong R_i$. The last proposition and the Remark 1.1.41 implies that

$$\mathscr{F}\big|_{D(f)} \cong \big(M_i \otimes_{R_i} R_f\big)^{\sim}.$$

Since X is compact, a finite number of such $D(f)$ will be sufficient. □

The proofs of the following propositions can be found in [Har77, II.5].

Proposition 1.1.53
Let X be a scheme and $\mathscr{F}$ an $\mathcal{O}_X$-module. Then $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} R$ of X, there exists an R-module M such that $\mathscr{F}\big|_U \cong \tilde{M}$. If X is noetherian, then $\mathscr{F}$ is coherent if and only if the same is true, with the extra condition that M is a finitely generated R-module.

Proposition 1.1.54
Let R be a ring and $X = \operatorname{Spec} R$. The functor which associates to every R-module M the quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ gives an equivalence of categories between the category of R-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. If R is noetherian, then the same functor gives an equivalence between finitely generated R-modules and coherent $\mathcal{O}_X$-modules.

Proposition 1.1.55
Let X be a scheme. The kernel, cokernel and image of any morphism of quasi-coherent sheaves is also quasi-coherent.

1.1.5 Sheaves of modules on a projective scheme

Let R be a graded ring, $(\operatorname{Proj} R, \mathcal{O})$ its spectrum and M be a graded R-module. We want to use M to define a sheaf of $\mathcal{O}$-modules $\tilde{M}$. We mimic here the construction of the structure sheaf of $\operatorname{Proj} R$. For an open set U of $\operatorname{Spec} R$, we set $\tilde{M}(U)$ as the set of all functions $s : U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})}$ such that:

- $s(\mathfrak{p}) \in M_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.

- For all $\mathfrak{p} \in U$ there exists an open neighbourhood V of $\mathfrak{p}$ contained in U and $m \in M$ and $r \in R$ two homogeneous elements of the same degree such that for each $\mathfrak{q} \in V$, $r \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{m}{r}$.

With the restriction as restriction maps, we get a sheaf. We define a structure of $\mathcal{O}(U)$-module on $\tilde{M}(U)$ as follows:

$$\begin{aligned} \mathcal{O}(U) \times \tilde{M}(U) &\longrightarrow \tilde{M}(U) \\ (f, s) &\longmapsto f \cdot s : \mathfrak{p} \longmapsto f(\mathfrak{p}) \cdot s(\mathfrak{p}), \end{aligned}$$

which is well-defined since M is a graded R-module. As restriction of functions, the restriction maps are compatible with the module structure.

Remark 1.1.56
When R is considered as a graded module over itself, $\tilde{R}$ is equal to the structure sheaf of $\operatorname{Proj} R$.

The following proposition is an updated version of Proposition 1.1.48.

Proposition 1.1.57
Let M be a graded R-module and let $X = \operatorname{Proj} R$. We have the following properties:

(i) *For every $\mathfrak{p} \in X$, $\tilde{M}_{\mathfrak{p}} \cong M_{(\mathfrak{p})}$.*

(ii) *For any homogeneous element $f \in R_+$, we have $\tilde{M}\big|_{D_+(f)} \cong \big(M_{(f)}\big)^{\sim}$, via the isomorphism $\big(D_+(f), \mathcal{O}\big|_{D_+(f)}\big) \cong \operatorname{Spec} R_{(f)}$.*

(iii) *$\tilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module. If R is noetherian and M is finitely generated, then $\tilde{M}$ is coherent.*

Definition 1.1.58
Let R be a graded ring, $X = \operatorname{Proj} R$ and $n \in \mathbb{Z}$. We define the $\mathcal{O}_X$-module $\mathcal{O}_X(n)$ by $\mathcal{O}_X(n) := R(n)^{\sim}$. If $\mathscr{F}$ is an $\mathcal{O}_X$-module, we denote by $\mathscr{F}(n)$ the $\mathcal{O}_X$-module $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathscr{F}$ by $\mathscr{F}(n)$. We call $\mathcal{O}_X(1)$ the twisting sheaf of Serre.

Proposition 1.1.59
Let R be a graded ring, $X = \operatorname{Proj} R$ and suppose that R is generated by R_1 as an R_0-algebra. Then:

(i) *The sheaf $\mathcal{O}_X(n)$ is an invertible sheaf for every $n \in \mathbb{Z}$.*

(ii) *For any R-module M and any $n \in \mathbb{Z}$, we have $\widetilde{M(n)} \cong \tilde{M}(n)$. In particular, $\mathcal{O}_X(n+m) \cong \mathcal{O}_X(n) \otimes \mathcal{O}_X(m)$.*

Proof. (i) It is sufficient to show that all the $\mathcal{O}_X(n)\big|_{D_+(f)}$ are free $\mathcal{O}_X\big|_{D_+(f)}$-modules of rank 1. Since R is generated by R_1 as an R_0-algebra, it is sufficient to do this for elements of R_1. So, let's take $f \in R_1$. The preceding proposition implies that $\mathcal{O}_X(n)\big|_{D_+(f)} \cong \widetilde{R(n)_{(f)}}$ as $\mathcal{O}_{\operatorname{Spec} R_{(f)}}$-module. Since $\partial f = 1$, the homomorphism of $R_{(f)}$-modules

$$\varphi : R_{(f)} \longrightarrow R(n)_{(f)}, \frac{r}{f^m} \longmapsto \frac{f^n r}{f^m}$$

is well-defined and is an isomorphism. Applying the functor $\sim$ to both sides of $R_{(f)} \cong R(n)_{(f)}$ gives $\widetilde{R(n)_{(f)}} \cong \mathcal{O}_{\operatorname{Spec} R_{(f)}}$, as required.

(ii) Since the Proposition 1.1.49.(*iii*) is still true in the projective case with our assumptions, we have

$$\tilde{M}(n) = \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \tilde{M} \otimes_{\mathcal{O}_X} \widetilde{R(n)} \cong M \widetilde{\otimes_R R(n)} \underset{A.1.9}{\cong} \widetilde{M(n)}$$

□

We will be especially interested in the case $R = k[x_0, \dots, x_n]$ for some field k. The next results will help us determine the global sections of $\mathcal{O}_X(n)$.

Definition 1.1.60 (R-module associated to a $\mathcal{O}_{\operatorname{Proj} R}$-module)
Let R be a graded ring, $X = \operatorname{Proj} R$ and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the graded R-module associated to a $\mathcal{O}_{\operatorname{Proj} R}$-module: we set

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n)),$$

as an abelian group. We want to define a structure of graded R-module on $\Gamma_*(\mathscr{F})$. Consider $r \in R_n$ and denote by $f_r \in \Gamma(X, \mathcal{O}_X(n))$ the map which send $\mathfrak{q}$ to $\frac{r}{1}$ for every $\mathfrak{q}$ in $\operatorname{Proj} R$. If $g \in \mathscr{F}(m)$, then consider $f_r \otimes g \in \Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathscr{F}(m))$ and use the isomorphism $\Gamma(X, \mathcal{O}_X(n)) \otimes \Gamma(X, \mathscr{F}(m)) \cong \mathscr{F}(m+n)$. Thus, we have a map

$$R_n \times \Gamma(X, \mathscr{F}(m)) \longrightarrow \Gamma(X, \mathscr{F}(m+n)).$$

With all these maps, $\Gamma_*(\mathscr{F})$ is a graded R-module.

Proposition 1.1.61
Let R be a ring, $r \in \mathbb{N}$, $S = R[x_0, \dots, x_r]$ and $X = \operatorname{Proj} S$. Then $S \cong \Gamma_*(\mathcal{O}_X)$, as graded modules.

Proof. See [Har77, Proposition II.5.13]. □

Corollary 1.1.62
Let R, X, r and S be as in the previous proposition. Let $n \in \mathbb{N}$. Then the global sections of $\mathcal{O}_X(n)$ are the homogeneous polynomials of degree n in $R[x_0, \dots, x_r]$.

Proof. We have:

$$\Gamma\big(X, \mathcal{O}_X(n)\big) \cong \Gamma\big(X, \widetilde{S(n)}\big) \underset{1.1.59}{\cong} \Gamma\big(X, \tilde{S}(n)\big) \cong \Gamma_*(S)_n \underset{1.1.61}{\cong} S_n.$$

□

Remark 1.1.63
This corollary implies that the global sections of $\mathcal{O}_X \cong \mathcal{O}_X(0)$ are R. This shows that if $r \neq 0$ and $n \neq 0$, then the invertible sheaf $\mathcal{O}_X(n)$ constructed is not isomorphic to $\mathcal{O}_X$ (and this is also an example of the fact that the stalks do not determine entirely a sheaf).

1.2 Divisors

1.2.1 Weil divisors

In this section, we omit certain definitions and most of proofs. The details can be found in II.6 of [Har77].

Definition 1.2.1 (Regular in codimension 1)
A scheme X is regular in codimension 1 *if all the rings $\mathcal{O}_{X,x}$ of dimension 1 are regular.*

Consider the following property:

($*$) X is a noetherian integral separated scheme which is regular in codimension one.

Unless specified otherwise, a scheme X satisfies ($*$).
The following proposition can be easily proved.

Proposition 1.2.2
Let U be an open set of X. If Y is a prime divisor of X, then $U \cap Y$ is a prime divisor of U. On the other hand, if Y is a prime divisor of U, then $\overline{Y}$ is a prime divisor of X. This correspondance is a $1-1$ correspondance between prime divisors of U and prime divisor of X which intersect U.

Proposition 1.2.3
Let $(X, \mathcal{O}_X)$ satisfy ($*$) and U be an open subset of X. Then U satisfies ($*$).

Proof. The scheme U is noetherian because localization of a noetherian ring is noetherian. Since an open set of an irreducible space is irreducible and since reducibility can be checked on the stalks, then U is irreducible and reduced, wich is equivalent to the integrality (see [Har77, II.3.1]). Since an open immersion is separated and composition of separated morphisms is again separated, so is U. Finally, since stalks are preserved by open immersion, then U is regular in codimension 1. $\square$

The following result will be used many times:

Proposition 1.2.4
Let X be a scheme. There is a bijection between integral closed subschemes of X and irreducible closed subsets of X.

The definitions and Theorem 6.2A of [Har77] implies that every local ring $\mathcal{O}_{X,x}$ of dimension 1 is a discrete valuation ring.

Example 1.2.5
Let k be an algebraic closed field. Then $\mathbb{P}_k^n$ satisfies ($*$). Denote by R the ring $k[x_0, \ldots, x_n]$. Since R is noetherian, R_+ has a finite set of generators $f_1, \ldots, f_s$. Thus, $\operatorname{Proj} R$ is covered by the set $D_+(f_i)$ and since $\mathbb{P}_k^n\big|_{D_+f_i} \cong \operatorname{Spec} R_{(f_i)}$ is noetherian, then so is $\mathbb{P}_k^n$. For all $\mathfrak{p} \in \operatorname{Proj} R$, we have $(\mathbb{P}_k^n)_\mathfrak{p} \cong R_{(\mathfrak{p})}$, which is reduced. Since $\operatorname{Proj} R$ is irreducible, then $\mathbb{P}_k^n$ is integral (Proposition 3.1 of [Har77]). Since $\operatorname{Proj} R$ is covered by open sets of the form $R_{(x_i)}$, then $\mathbb{P}_k^n$ is regular in codimension one. Separatedness can be checked on the sets $D_+(x_i)$.

Definition 1.2.6 (Prime divisor)
A prime divisor *on X is a closed integral subscheme Y of codimension 1.*

Example 1.2.7
If $X = \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, for some algebraically closed field, then X satisfies $(*)$. Furthermore, a prime divisor of X is given by $\mathcal{V}(f)$ for an irreducible polynomial f. Indeed, a prime divisor Y of X is given by $Y = \mathcal{V}(I)$ for some ideal I of $k[x_1, \dots, x_n]$. Since Y is irreducible, I must be a prime ideal. The dimension of Y implies that the height of I is 1 and and so $I = \langle f \rangle$ for some irreducible polynomial (see I.1.8A of [Har77] and Theorem 47, section 19 of [Mat70]).

Definition 1.2.8 (Weil divisor)
Let $\operatorname{Div} X$ *be the free abelian group with basis*

$$\big\{Y : Y \text{ prime divisor of } X\big\}.$$

Then, a Weil divisor D *of* X *is an element* $D = \sum n_i Y_i$ *of* $\operatorname{Div} X$*, where only finitely many* n_i *are non-zero.*

Proposition 1.2.9
Let Z *be an irreducible scheme. Then, the topological space* Z *contains a unique point* η *such that* $\overline{\{\eta\}} = Z$*. This point is called the* generic point *of* Z*.*

Proof. If $Z = \operatorname{Spec} R$ is affine, take η as the nilradical of R (which is prime since $\operatorname{Spec} R$ is irreducible). If Z is an arbitrary scheme, take an affine subset V. Then V contains a generic point η and since V is dense in U, then η is a generic point of Z. The unicity of the generic point comes from the irreducibility of Z. □

Proposition 1.2.10
Let X *be an integral scheme and let* η *be the generic point of* X*. Then,* $\mathcal{O}_{X,\eta}$ *is a field.*

Proof. Let U be a non-empty affine subset of X (which must contain η). Since U is irreducible U contains a unique generic point which must be η. On the other hand, we have $U \cong \operatorname{Spec} R$ for some integral domain R. Thus, η corresponds to the ideal 0 in R. Finally, we get

$$\mathcal{O}_{X,\eta} \cong \big(\mathcal{O}_X\big|_U\big)_\eta \cong R_0,$$

as required. □

Definition 1.2.11 (Field of rational functions)
Let X *be an integral scheme and let* η *be its generic point. The field* $\mathcal{O}_{X,\eta}$ *is called the* field of rational functions *of* X*. We denote it by* $K(X)$ *(or* K *if it is clear from the context).*

Proposition 1.2.12
Let X *be as above and* U *be an open affine subset of* X*. Then, the field of fractions of* $\mathcal{O}_X(U)$ *is equal to* K*.*

Proof. The field of fractions of $\mathcal{O}_X(U)$ is equal to the field of fractions of the ring R, where R is such that $U \cong \operatorname{Spec} R$. Therefore

$$\operatorname{Frac}\big(\mathcal{O}_X(U)\big) \cong \operatorname{Frac}(R) = R_0 = K.$$

□

Proposition 1.2.13
Let X be an integral scheme and let U be an open set of X. For every $x \in U$, the canonical map $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x}$ is injective.

Proof. Let U and x and suppose $s \in \mathcal{O}_X(U)$ is such that $s_x = 0$. This means that $s\big|_V$ for some neighbourhood V of x. Now, consider any affine open set $W \subset U$. We have the homomorphisms

$$\mathcal{O}_X(W) \hookrightarrow \mathrm{Frac}\mathcal{O}_X(W) \xrightarrow[\cong]{} \mathcal{O}_{X,\eta}.$$

Since $\eta \in V \cap W$, the image of $s\big|_W$ in $\mathcal{O}_{X,\eta}$ will be zero, which implies that $s\big|_W = 0$. Covering U with affine subsets gives $s = 0$, as required. □

Corollary 1.2.14
Let X be an integral scheme. The canonical map $\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,\eta}$ is injective.

Proof. Let $[U, s] \in \mathcal{O}_{X,x}$ such that $s_\eta = 0$. Applying the previous result to the morphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\eta}$ gives $s = 0$. □

Lemma 1.2.15
Let R be a commutative ring and $\mathfrak{p} \in \operatorname{Spec} R$. Then, there is a bijection between $\operatorname{Spec} R_\mathfrak{p}$ and the set of prime ideals of R which are contained in $\mathfrak{p}$.

Proof. Follows directly from the well-known result: if S be a multiplicative subset of a commutative ring R, then there is a bijection between $\operatorname{Spec} S^{-1}R$ and the set of prime ideals of R which do not intersect S. □

Proposition 1.2.16
Let X be an arbitrary scheme and $x \in X$ a point. Then, there is a $1-1$ correspondance between the following sets:

(i) *The set of irreducible closed sets which contain x.*

(ii) *The set of points $z \in X$ such that $x \in \overline{\{z\}}$ (we say that z specializes to x).*

(iii) $\operatorname{Spec} \mathcal{O}_{X,x}$.

Proof. $(i) \leftrightarrow (ii)$ Let Z be an irreducible closed subset which contains x and denote by η_Z the generic point of Z. Then, $x \in \overline{\{\eta_Z\}} = Z$. On the other hand, if $z \in X$ is such that $x \in \overline{\{z\}} =: Z$, then z is the generic point of the integral subscheme Z of X.

$(ii) \leftrightarrow (iii)$ First, suppose that $z \in X$ is such that $x \in \overline{\{z\}}$ and choose an open neighbourhood U of x. Since U is open, we must have $z \in U$. Hence, we can fix an affine neighbourhood U of x and work in U. We have

$$\operatorname{Spec} \mathcal{O}_{X,x} \xrightarrow{\cong} \operatorname{Spec} R_\mathfrak{p} \longrightarrow \operatorname{Spec} R \xrightarrow{\cong} U,$$

where $\mathfrak{p}$ is the prime ideal corresponding to x. Using the previous lemma, we have:

$$\operatorname{Spec} R_p \overset{1-1}{\longleftrightarrow} \left\{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{q} \subset \mathfrak{p}\right\} \overset{1-1}{\longleftrightarrow} \left\{\mathfrak{q} \in \operatorname{Spec} R : \mathfrak{p} \in \overline{\{\mathfrak{q}\}}\right\},$$

as required.

□

Corollary 1.2.17
Let X be a scheme which satisfies $(*)$ and $x \in X$ a point. Then, we have a injection from the set of prime divisor passing through x to the set of prime divisor of $\operatorname{Spec} \mathcal{O}_{X,x}$.

Proof. As in the proof of the proposition, we fix an affine neighbourhood U of x and denote by $\mathfrak{p}$ the point in $\operatorname{Spec} R$ corresponding to x. If Z is a divisor of X passing through x, we denote by $\mathfrak{q}$ its image in $\operatorname{Spec} R$ (we then have $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{p}$ is the point corresponding to the generic point of Z). Now, $\overline{\{\mathfrak{q}_\mathfrak{p}\}}$ is a closed irreducible subset of $\operatorname{Spec} R_\mathfrak{p}$. Suppose that $\mathfrak{q}'_\mathfrak{p}$ is a proper closed irreducible subset of $R_\mathfrak{p}$ containing $\mathfrak{q}_\mathfrak{p}$, we get a point z' such that $\overline{\{z\}} \subset \overline{\{z'\}}$. Since Z is of codimension 1, we have $z = z'$ and so $\mathfrak{q}'_\mathfrak{p} = \mathfrak{q}_\mathfrak{p}$, as required. □

Proposition 1.2.18
If X is an irreducible scheme and Y is an integral closed subscheme with generic point η, then $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Y, X)$. *In particular, if X satisfies $(*)$ and Y has codimension* 1, *then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring.*

Let Y be a prime divisor of X and $\eta \in Y$ be its generic point. Then, the local ring $\mathcal{O}_{X,\eta}$ is a valuation ring with valuation v_Y. Furthermore, the quotient field K of $\mathcal{O}_{X,\eta}$ is the field of functions of X and the valuation v_Y extends to a valuation, again denoted v_Y, on K.

Definition 1.2.19 (Zero, pole)
Let $f \in K^*$ be a rational function on X. If $v_Y(f) > 0$, we say that f has a zero along Y of order $v_Y(f)$; if $v_Y(f) < 0$, we say that f has a pole of order $v_Y(f)$ along Y.

Proposition 1.2.20
Let X satisfy $(*)$ and let $f \in K^*$. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y of X.

Definition 1.2.21 (Principal divisor)
Let $f \in K^*$ and consider the finite sum

$$(f) = \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisor Y of X. Then (f) is called the principal divisor associated to f

Let $f, g \in K^*$. By the properties of the valuations, we have $\left(\frac{f}{g}\right) = (f) - (g)$. Hence, we get an homomorphism Φ_{Div} from K^* to the group $\operatorname{Div} X$.

Example 1.2.22
Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ for some algebraically closed field and $f \in k[x,y]$ be a curve. Then the vanishing set Z of f can be written as $Z = \mathcal{V}(f_1) \cup \ldots \cup \mathcal{V}(f_n)$, where the f_i are irreducible polynomials. Then for $P = \langle f_i \rangle$, $v_P(f)$ is the integer r such that $f \in P_P^r$ and $f \notin P_P^{r+1}$. For example, take $f(x,y) = x^2 y$. Its vanishing set splits into $V(x) \cup V(y)$. We see that $v_{\langle x \rangle} = 2$ and $v_{\langle y \rangle} = 1$. Therefore, $(f) = 2 \cdot \mathcal{V}(x) + \mathcal{V}(y)$. In a similar way, we get $\left(\frac{x^2}{y}\right) = 2 \cdot \mathcal{V}(x) - \mathcal{V}(y)$. We can write this in an other way: if f is as above, then $f = f_1 \cdot \ldots \cdot f_n$ for some irreducible polynomials f_i and $v_{\langle f_i \rangle}(f)$ is the maximal power r such that f_i^r divides f.

Definition 1.2.23 (Group of principal divisors, divisor class group)
The group of principal divisors *is the image of the homomorphism mentioned above. The* divisor class group*, which is denoted* $\operatorname{Cl} X$ *is the quotient of* $\operatorname{Div} X$ *by the subgroup of principal divisors.*

Example 1.2.24
We have $\operatorname{Cl} \mathbb{A}_k^n = 0$. Indeed, let $\sum n_i Y_i$ a divisor of $\mathbb{A}_k^n$ and write $Y_i = \mathcal{V}(f_i)$ for some irreducible polynomial f_i (see Example 1.2.7). Then we have $\Phi_{\mathrm{Div}}\left(\prod f_i^{n_i}\right) = \sum n_i Y_i$.

1.2.1.1 The case of projective space

We consider in this section $X = \mathbb{P}_k^n$. We saw above that the scheme X satisfy $(*)$. Let Y be a prime divisor of X. As in the affine case, we have $Y = \mathcal{V}(f)$ for some irreducible homogeneous polynomial f. We define a equivalence relation on the set of irreducible homogeneous polynomials of $k[x_0, \ldots, x_n]$:

$$f \sim g \Leftrightarrow \exists \lambda \in k^* \text{ such that } f = \lambda \cdot g.$$

Then, we get a bijection between the set of equivalences classes and the prime divisors of X.

Definition 1.2.25 (Degree of a divisor)
Let $Y = \sum n_i Y_i$ *be a Weil divisor of* Y*. We know that there exists a finite collection of polynomials* $f_1, \ldots, f_n$ *such that* $Y = \sum n_i \mathcal{V}(f_i)$*. We define the* degree *of* Y *as* $\sum_{i=1}^n n_i \cdot \partial f_i$*. This degree is denoted* ∂Y*.*

Remark 1.2.26
This notion is well-defined since $f \sim g$ implies $\partial f = \partial g$.

Remark 1.2.27
Let h be an homogeneous polynomial. If $h = f \cdot g$ for some polynomials f and g, then both f and g are homogeneous.

We want to look at the principal divisors of X. As above, we consider some non-zero homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ and write it $f = f_1 \cdot \ldots \cdot f_n$ where the f_i's are homogeneous irreducible polynomials. Then, then divisor (f) associated to f is $\sum n_i \mathcal{V}(f_i)$ where $n_i = \max\left\{k \in \mathbb{N} : f_i^k \mid f\right\}$. As above, we have $\left(\frac{f}{g}\right) = (f) - (g)$.

We denote by H the prime divisor $x_0 = 0$: $H = \mathcal{V}(x_0)$.

Proposition 1.2.28
Let Y *be a prime divisor of degree* d *of* X*. Then* $Y \sim dH$*.*

Proof. Let $f_1, \ldots, f_n$ denote polynomials such that $Y = \sum a_i \mathcal{V}(f_i)$ and set $f = f_1^{a_1} \cdot \ldots \cdot f_n^{a_n}$. We can write $f = \frac{g_1^{b_1} \cdot \ldots \cdot g_r^{b_r}}{h_1^{c_1} \cdot \ldots \cdot h_r^{c_s}}$ where every b_i and c_j is greater than 0. By hypothesis, the degree of the numerator minus the degree of the denominator is equal to d. Hence, we have

$$\left(\frac{g_1^{b_1} \cdot \ldots \cdot g_r^{b_r}}{h_1^{c_1} \cdot \ldots \cdot h_r^{c_s} \cdot x_0^d}\right) = Y - dH,$$

as required. □

Proposition 1.2.29
Let $f \in K^*$. Then, $\partial(f) = 0$.

Proof. We know that f can be written as a quotient of two homogeneous polynomials of the same degree. Factor both and them to get (f) and the degree will be 0. □

Proposition 1.2.30
The degree map $\partial : \operatorname{Div} X \longrightarrow \mathbb{Z}$ yields to an isomorphism $\operatorname{Cl} X \cong \mathbb{Z}$.

Proof. It is clear that the degree if a homomorphism of groups and it is surjectivce because $\partial(dH) = d$ for all $d \in \mathbb{Z}$. The last proposition shows that principal divisors are in the kernel of ∂. Now, if $D = \sum n_i Y_i - \sum n_j' Y_j'$, with $n_i, n_j' \geq 0$ and Y_i, Y_j' prime, has degree 0, we have

$$D \sim \sum n_i \partial Y_i \cdot H - \sum n_j' \partial Y_j' \cdot H = \sum (n_i \partial Y_i \cdot H - \sum n_j' \partial Y_j') \cdot H = 0 \cdot H.$$

Hence, D is of the form $D = (f)$, as required. □

We saw above (Proposition 1.1.59) that the $\mathcal{O}_X(n)$ are invertible $\mathcal{O}_X$-modules on X. One can ask if they are all the invertible sheaves (that is if the group of invertible $\mathcal{O}_X$-modules on X is isomorphic to $\mathbb{Z}$). We will see below that there is some connection between the divisor class group and the group of invertible sheaves.

1.2.2 Cartier divisors

Proposition 1.2.31
Let $\mathscr{F}$ be a sheaf of rings. Then, the sheaf $\mathscr{F}^*$ which associate to every open set U the group $\mathscr{F}^*(U)$ is a sheaf of groups.

Definition 1.2.32 (Regular element)
Let R be a ring. We say that $r \in R \setminus \{0\}$ is a regular element *of R if it is neither a left nor a right zero divisor.*

Remarks 1.2.33 (i) Invertible elements are regular elements.

(ii) Suppose now that R is commutative. Then, the set S of all regular elements of R is a multiplicative subset of R. Furthermore, it is the biggest set such that $\varphi : R \longrightarrow S^{-1}R$ is injective.

Let X be a scheme. For every open set U of X, we define:

$$S(U) = \{ s \in \mathcal{O}_X(U) : s_x \text{ is regular } \forall x \in U \}.$$

We easily check that $S(U)$ is a multiplicative subset of $\mathcal{O}_X(U)$.

Definition 1.2.34 (Sheaf of total quotient rings)
Let X be a scheme. We call sheaf of total quotient rings *the sheaf associated to the presheaf $U \longmapsto S^{-1}(U)\mathcal{O}_X(U)$. This sheaf is denoted by $\mathscr{K}$.*

Remarks 1.2.35 (i) One could have defined the set $S(U)$ as the set of regular elements of $\mathcal{O}_X(U)$. The problem is that a regular element s can be sent to a zero divisor by the restriction map and the association $U \longmapsto S^{-1}(U)\mathcal{O}_X(U)$ fails to be a presheaf.

(ii) Since $\mathcal{O}_X$ can be identified to a subring of $S^{-1}(U)\mathcal{O}_X(U)$, we obtain an injective morphism $i : \mathcal{O}_X \longrightarrow \mathscr{K}$. In a similar way, we get a monomorpism $\mathcal{O}_X^* \longrightarrow \mathscr{K}^*$.

Definition 1.2.36 (Cartier divisor)
A Cartier divisor *of a scheme X is a global section of $\mathscr{K}^*/\mathcal{O}_X^*$. We denote by* CaDiv$(X)$ *the group of Cartier divisors of X.*

Definition 1.2.37 (Principal Cartier divisor, linearly equivalent Cartier divisors)
A Cartier divisor is principal *if it is in the image of π_X, where $\pi : \mathscr{K}^* \longrightarrow \mathscr{K}^*/\mathcal{O}_X^*$ is the canonical map. Two cartiers divisors f, g are* linearly equivalent *if f/g is principal. We denote by* CaCl(X) *the group of Cartier divisor modulo principal divisors.*

To describe a Cartier divisor, we can give an open covering $\{U_i\}$ of X and elements $f_i \in \mathscr{K}^*(U_i)$ which can be glued together. This is equivalent to say that the element $f_i|_{U_i \cap U_j} \cdot f_j|_{U_i \cap U_j}{}^{-1}$ is in $\mathcal{O}_X^*(U_i \cap U_j)$ for all i, j. In this case, the multiplication of two Cartier divisors is given by

$$\{U_i, f_i\} \cdot \{U'_j, f'_j\} = \left\{U_i \cap U'_j, f_i|_{U_i \cap U'_j} \cdot f'_j|_{U_i \cap U'_j}\right\}.$$

1.2.2.1 The case of an integral separated scheme

When we define a presheaf of abelian groups on a topological space X, we said that it is a contravariant functor from the category of open sets of X (which is denoted $\mathscr{T}_X$) to the category of abelian groups (with the extra condition that the empty set is mapped to the trivial group). We could have chosen some subcategory of $\mathscr{T}_X$ and tried to do the same. For example, let $\mathcal{C}$ be some base for the topology of X closed under intersection (considered as a category with inclusions as morphisms). We can define presheaves of $\mathcal{C}$. Furthermore, we can express sheaves conditions as we did in the "standard case". Now, if $\mathscr{F}_\mathcal{C}$ is a sheaf on X (i.e., a functor from $\mathcal{C}$ to the category of abelian groups, or rings, ...) there exists a unique sheaf $\mathscr{F}$ which extend $\mathscr{F}_\mathcal{C}$, that is $\mathscr{F}_\mathcal{C}(V) = \mathscr{F}(V)$ for every $V \in \mathcal{C}$.

Proposition 1.2.38
Let X be a separated scheme and let U, V be two affine sets. Then, $U \cap V$ is affine.

Proof. See Chapter 3, Proposition 3.15 of [Liu06]. □

Proposition 1.2.39
Let X be a integral separated scheme. Then, $\mathscr{K}$ is the constant sheaf K, where K is the field of functions of X.

Proof. We know (cf. Proposition 1.2.12) that the field of fractions of $\mathcal{O}_X(U)$ is K for every affine set U. Taking $\mathcal{C}$ to be the basis consisting of all open affine sets, we get the constant presheaf $\mathscr{F}_\mathcal{C}$ which maps every affine set to K. Since X is irreducible, so is any open subset U of X. Hence, any open set is connected and the presheaf $\mathscr{F}_\mathcal{C}$ is a sheaf. Therefore, the sheaf of total quotient rings $\mathscr{K}$ is the unique sheaf which extends $\mathscr{F}_\mathcal{C}$ (see the remark above). Hence, $\mathscr{F}(U) = K$ for every open set U. □

Example 1.2.40 (The case of the projective space)
We take $X = \mathbb{P}^n_k$. We see that K is the subfield of $k(x_0, \ldots, x_n)$ consisting of elements of degree 0 and for $0 \leq i < j \leq n$:

$$\begin{aligned} \mathcal{O}_X(U_i \cap U_j)^* \cong k[x_0, \ldots, x_n]^*_{(x_i x_j)} &= \left\{ f \in k\left[x_0, \ldots, x_n, \frac{1}{x_i}, \frac{1}{x_j}\right] : \partial f = 0 \right\}^* \\ &= \big\{ f \in k(x_i, x_j) : \partial f = 0 \big\}. \end{aligned}$$

Hence, a Cartier divisor can be given by a collection of polynomials $f_0, \ldots, f_n$ in $k(x_0, \ldots, x_n)$ such that $f_i {f_j}^{-1} \in k(x_i, x_j)$. For example, choose an homogeneous polynomial f of degree 1 in $k[x_0, \ldots, x_n]$ and set $f_i = \frac{f}{x_i}$. Then, $\big\{D_+(x_i), f_i\big\}$ is a Cartier divisor.

1.2.2.2 Links between Weil and Cartier divisors

Definition 1.2.41 (Locally factorial scheme)
We say that a scheme X is locally factorial *if $\mathcal{O}_{X,x}$ is a UFD for every $x \in X$.*

Proposition 1.2.42
Let X be a locally factorial scheme which satisfies $(*)$, that is: X is a noetherian integral separated scheme which is regular in codimension one.Then, the group $\operatorname{Div} X$ *is isomorphic to the group of cartier divisor $\mathscr{K}^*/\mathcal{O}^*_X(X)$. Furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Proof. First, remark that a Cartier divisor can be given in many ways, depending on the open covering we choose. Hence, we can fix an open affine covering $\{U_i\}_{i \in I}$ of X (since X is noetherian, we may assume that I is finite). Now, a Cartier divisor is a collection of elements $f_i \in \mathscr{K}^*(U_i)$ such that $\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \mathcal{O}^*_X(U_i \cap U_j)$ for all i, j. Proposition 1.2.12 implies that $\mathscr{K}^*(U_i) \cong K^*$ so we may assume that each f_i lives in K^*. Let f be the global section defined by the $\{U_i, f_i\}$ and Y be a prime divisor of Y. Then, pick some i such that $Y \cap U_i \neq \emptyset$ and set $n_Y(f) = v_Y(f_i)$. If i and j are such that $U_i \cap Y \neq \emptyset \neq Y \cap U_j$, then the condition $\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \mathcal{O}^*_X(U_i \cap U_j)$ implies $v_Y\left(\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}}\right) = 0$ and thus $v_Y(f_i) = v_Y(f_j)$. Hence, we get a divisor $\sum n_Y(f) \cdot Y$ (remark that only a finite number of the $v_Y(f_i) \neq 0$ for each f_i, so the sum is finite since the number of the U_i is also finite). Note that the properties of the valuations imply that the association

$$\Psi : f \longmapsto \sum n_Y(f) \cdot Y$$

is a homomorphism of groups.

Now, let Y be a prime divisor on X. For every $x \in X$, we denote by U_x an affine open set which contains x. If Y contains x, we get a prime divisor Y_x of $\operatorname{Spec} \mathcal{O}_{X,x}$ (see Corollary 1.2.17). Since $\mathcal{O}_{X,x}$ is an UFD, there exists some $f'_x \in \operatorname{Frac} \mathcal{O}_{X,x}$ such that $(f'_x) = Y_x$ (the proof is similar as in the Example 1.2.24; see Proposition II.6.2 of [Har77]). Consider the image of f'_x in K under the injective morphism $\operatorname{Frac} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,\eta}$ (see Corollary 1.2.14) and call it g'_x. We get a divisor (g'_x) of X and its image in $\operatorname{Spec} \mathcal{O}_{X,x}$ is equal to Y_x. Thus, Y and (g'_x) have the same "Y-component", that is: $(g'_x) = Y + \sum_{i=1}^n n_i \cdot Y_i$ with $x \notin Y_i$. Suppose that $f'_x = [V_x, f_x]$

where $f_x \in \operatorname{Frac} \mathcal{O}_X(V_x)$ and take $W_x \subset V_x \bigcap_{i=1}^n (X \setminus Y_i)$ with W_x affine. Then, the element $f_x = f'_x\big|_{W_x}$ satisfies $(f_x)\big|_{W_x} = Y\big|_{W_x}$ (if f_x is viewed in K). Since X is compact, we can cover X with the W_x (if $x \notin Y$, just take $f_x = 1$). These elements can be patched together to get a Cartier divisor $\{W_x, f_x\}$ (see Lemma 32 of [Mur06]). The properties of the free abelian group imply the existence of a homomorphism $\Phi : \operatorname{Div}(X) \longrightarrow \operatorname{CaDiv}(X)$.

To check that $\Psi\Phi = \mathrm{id}_{\operatorname{Div} X}$, it suffice to show that $\Psi\Phi(Y) = Y$ for every prime divisor $Y \in \operatorname{Div}(X)$. If $\{U_x, f_x\} = \Phi(Y)$, then $(f_x)\big|_{U_x} = Y\big|_{U_x}$ for every x such that $U_X \cap Y \neq \emptyset$. Thus, $v_Y(f_x) = 1$. If Z is another prime divisor, there exists some U_x such that $U_x \cap Z \neq \emptyset$ and $v_Z(f_x) = 0$ which implies that Z have coefficient zero in $\Psi\Phi(Y)$.

Now, if $C = \{U_x, f_x\}$ is a Cartier divisor and $D = \sum C_Y \cdot Y$ is obtained from D, set g_x as the image in the field of function of $\operatorname{Spec} \mathcal{O}_{X,x}$ of the element f_x. Then, $(g_x) = D_x$ and the image of D in $\operatorname{CaDiv}(X)$ will be C.

Finally, it is easy to see that principal Cartier divisors correspond to principal Weil divisors: if $C = \pi(f)$, then the Weil divisor $\Psi(C)$ is (f). On the other hand, if $f \in K$ gives the Weil divisor D, then the image $g \in \mathscr{K}^*$ of f will give the Cartier divisor $\Phi(D)$. □

Example 1.2.43
We come back to the Example 1.2.40 and take an homogeneous polynomial f of degree 1 in $k[x_0, \ldots, x_n]$. We have the Cartier divisor C represented by $\big\{D_+(x_i), f_i\big\}$. On the other hand, f define a Weil prime divisor $\mathcal{V}(f)$. These two divisors are associated under the bijection between $\operatorname{Div}(X)$ and $\operatorname{CaDiv}(X)$.

1.2.3 Picard group

Receall (Definition 1.1.34) that an $\mathcal{O}_X$-module $\mathscr{F}$ is *invertible* if it is locally free of rank 1.

Proposition 1.2.44
The product of two invertible sheaves is again invertible.

Proof. Let $\{V_i\}$ be an open covering of X such that $\mathscr{F}\big|_{V_i}$ and $\mathscr{G}\big|_{V_i}$ are free of rank 1. Then,

$$\big(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}\big)\big|_{V_i} \cong \mathscr{F}\big|_{V_i} \otimes_{\mathcal{O}_X|_{V_i}} \mathscr{G}\big|_{V_i} \cong \mathcal{O}_X\big|_{V_i} \otimes_{\mathcal{O}_X|_{V_i}} \mathcal{O}_X\big|_{V_i} \cong \mathcal{O}_X\big|_{V_i}.$$

□

Proposition 1.2.45
Let $\mathscr{F}$ be an invertible sheaf. Then, the sheaf $\mathscr{H}\mathrm{om}(\mathscr{F}, \mathcal{O}_X)$ satisfies

$$\mathscr{H}\mathrm{om}(\mathscr{F}, \mathcal{O}_X) \otimes \mathscr{F} \cong \mathcal{O}_X.$$

Sketch of proof. First, show that $\mathscr{H}\mathrm{om}(\mathscr{F}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathscr{F} \cong \mathscr{H}\mathrm{om}(\mathscr{F}, \mathscr{F})$. Then, the result follows since $\operatorname{Hom}\big(\mathcal{O}_X\big|_U, \mathcal{O}_X\big|_U\big) \cong \mathcal{O}_X(U)$. □

Corollary 1.2.46
Let $\mathcal{O}_X$ be a ringed space. The set of isomorphism classes of invertible $\mathcal{O}_X$-modules with $\otimes$ is an abelian group.

Definition 1.2.47 (Picard group)
The group defined in the last Corollary is the Picard group *of the scheme X. It is denoted by* $\operatorname{Pic}(X)$.

Example 1.2.48
We saw that the $\mathcal{O}_X(m)$ are invertible sheaves on $\mathbb{P}^n$. Thus, $\operatorname{Pic}\left(\mathbb{P}^n\right)$ contains a subgroup which is the image of $\mathbb{Z}$. On the other hand, we saw that $\operatorname{Cl}\mathbb{P}^n \cong \mathbb{Z}$. We will see later that we have $\operatorname{Pic} X \cong \operatorname{Cl} X$.

1.2.3.1 Links between Cartier divisors and the Picard group

Let X be a scheme and let D be a Cartier divisor on X represented by $\{U_i, f_i\}$. For each i, let $\mathscr{F}_i$ be the $\mathcal{O}_X\big|_{U_i}$-module defined as follows: for all $U \subset U_i$, set

$$\mathscr{F}_i(U) = \mathcal{O}_X(U) \cdot f_i^{-1}\big|_U.$$

Then, the map which send $r \in \mathcal{O}_X(U)$ to $rf_i^{-1}\big|_U$ is an isomorphism. The surjectivity is clear. Now, if we write $f_i\big|_U$ as $\frac{r'}{s'}$ and r is such that $rf_i^{-1}\big|_U$, we get $rs's'' = 0$ for some $s'' \in S(U)$ which implies that $r = 0$. Since $\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \mathcal{O}_X^*(U_i \cap U_j)$, the elements $f_i\big|_{U_i \cap U_j}$ and $f_j\big|_{U_i \cap U_j}$ generate the same $\mathcal{O}_X(U_i \cap U_j)$ module. Therefore, we can glue the $\mathscr{F}_i$ to get an invertible $\mathcal{O}_X$-module $\mathcal{L}(D)$. Note that $\mathcal{L}(D)$ is a subsheaf of $\mathscr{K}$.

Remark 1.2.49
If we take each $f_i = 1$, we have $\mathcal{L}(D)(U_i) = \mathcal{O}_X(U_i)$ and so $\mathcal{L}(D) \cong \mathcal{O}_X$. Hence, the neutral element of $\operatorname{CaDiv}(X)$ is sent to the neutral element of $\operatorname{Pic}(X)$.

Proposition 1.2.50
The map which send a Cartier divisor D to $\mathcal{L}(D)$ gives rise to a bijection between Cartier divisors on X and invertible subsheaves of $\mathscr{K}$.

Proof. Let $\mathcal{L}$ be an invertible subsheaf of $\mathscr{K}$. Choose a generator $f \in \mathcal{O}_X(X)$ (which means that $f\big|_U$ is a generator for $\mathcal{L}(U)$ as an $\mathcal{O}_X(U)$-module for each open set U). Then, it is easy to see that $\mathcal{L}\big(\{X, f^{-1}\}\big) = \mathcal{L}$. Let $\mathcal{L}$ be an invertible subsheaf of $\mathscr{K}$. If $\{f_i\}$ and $\{f_i'\}$ are two family of elements such that f_i^{-1} and $f_i'^{-1}$ both generate $\mathcal{L}$, then f_i must differ from f_i' by an invertible element of $\mathcal{O}_X(U_i)$. Hence, the global sections in $\mathscr{K}^*/\mathcal{O}_X^*$ will be the same. □

Proposition 1.2.51
Let D_1 and D_2 be two Cartier divisors. Then, $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ (as $\mathcal{O}_X$-modules).

Proof. We can make two proofs of this fact:

(i) First, note that

$$\mathcal{L}(D_1) \cong \mathcal{L}(D_2) \Leftrightarrow \mathcal{L}(D_1) \otimes_{\mathcal{O}_X} \mathcal{L}(D_2)^{-1} \cong \mathcal{O}_X \Leftrightarrow \mathcal{L}(D_1/D_2) \cong \mathcal{O}_X.$$

Hence, it suffices to show that a Cartier divisor D is principal if and only if $\mathcal{L}(D) \cong \mathcal{O}_X$. Suppose D is principal and let $f \in \mathscr{K}^*$ be such that $D = \pi_X(f)$, where $\pi : \mathscr{K}^* \longrightarrow \mathscr{K}^*/\mathcal{O}_X^*$. Then, $D = \{X, f\}$ and $f\big|_U$ is a basis for $\mathcal{L}(D)(U)$ for every open set U. On the other hand, if $\mathcal{L}(D) \cong \mathcal{O}_X$, then choose a basis $\{f\}$ of $\mathcal{L}(D)(X)$. Now, we have $D = \pi_X(f)$, as required.

(ii) We have the exact sequence $1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathscr{K}^* \xrightarrow{\pi} \mathscr{K}^*/\mathcal{O}_X^* \longrightarrow 1$ which gives rise to a long exact sequence in cohomology

$$1 \longrightarrow \mathcal{O}_X^*(X) \xrightarrow{\eta} \mathscr{K}^*(X) \xrightarrow{\pi_X} \mathscr{K}^*/\mathcal{O}_X^*(X) \xrightarrow{\delta^0} H^1\big(\mathcal{O}_X^*\big) \longrightarrow \dots$$

Since $\mathrm{Pic}(X) \cong H^1\big(X, \mathcal{O}_X^*\big)$ (see Theorem 2.3.3), the result follows.

□

Using the above results we get the following proposition:

Proposition 1.2.52
Let X be an integral scheme. Then, the map $\psi : \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(\mathcal{O}_X)$ which sends D to $\mathcal{L}(D)$ is bijective.

Proof. By the previous two propositions we know that ψ is injective. Thus, all we have to show is that we can realise an inversible sheaf as a subsheaf of $\mathscr{K}$.
Let $\mathcal{L}$ be an invertible sheaf on X. We have to define a monomorphism $\alpha : \mathcal{L} \longrightarrow \mathscr{K}$. If $\eta \in X$ denote the generic point of X, the fact that X is integral implies

$$\mathscr{K}(U) \cong K \cong \mathcal{O}_{X,\eta}.$$

Thus, we can define:

$$\begin{aligned} \alpha_U : \mathcal{L}(U) &\longrightarrow \mathcal{O}_{X,\eta} \\ s &\longmapsto s_\eta. \end{aligned}$$

On the stalks, we have $\alpha_x\big([U,s]\big) = [U,s] \in \mathcal{O}_{X,\eta}$. This implies that α_x is a monomorphism (see Corollary 1.2.14) and so is α. □

Proposition 1.2.53
Let X be a locally factorial scheme which satisfies $(*)$. Then, $\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$.

Proof. Follows from the last proposition and Proposition 1.2.42. □

Proposition 1.2.54
Let $X = \mathbb{P}_k^n$. Then, every invertible sheaf $\mathcal{L}$ is isomorphic to some $\mathcal{O}_X(m)$, for a unique $m \in \mathbb{Z}$.

Proof. We have the following isomorphisms:

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\cong} & \mathrm{Cl}(X) & \xrightarrow{\cong} & \mathrm{CaCl}(X) & \xrightarrow[\varphi]{\cong} & \mathrm{Pic}(X) \\ 1 & \longmapsto & \mathcal{V}(x_0) & \xmapsto{1.2.43} & \Big[\big\{D_+(x_i), \frac{x_0}{x_i}\big)\Big] & \xmapsto[\varphi]{} & \mathcal{O}_X(1) \end{array}$$

We only have to check that the image of $\Big[\big\{D_+(x_i), \frac{x_0}{x_i}\big)\Big]$ by φ is indeed $\mathcal{O}_X(1)$.
We denote by f_{ij} the element $f_i|_{U_i \cap U_j} \cdot \big(f_j|_{U_i \cap U_j}\big)^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$ of the association from Cartier divisor to the Picard group.

(i) For $\mathcal{O}_X(1)$
We know that $\mathcal{O}_X(1)\big(D_+(x_i)\big)$ can be identified with polynomials of degree 1 in $k\big[x_0, \dots, x_n, \frac{1}{x_i}\big]$. Hence, we have $\mathcal{O}_X(1)\big(D_+(x_i)\big) = x_i \cdot \mathcal{O}_X\big(D_+(x_i)\big)$ which implies that $f_{ij} = \frac{x_j}{x_i}$.

(ii) For the Cartier divisor $\left[\left\{D_+(x_i), \frac{x_0}{x_i}\right)\right]$
We have $f_{ij} = \frac{x_0}{x_i}\left(\frac{x_0}{x_j}\right)^{-1} = \frac{x_j}{x_i}$.

This implies that the image of $\left[\left\{D_+(x_i), \frac{x_0}{x_i}\right)\right]$ in $\mathrm{Pic}(X)$ is $\mathcal{O}_X(1)$. Hence, $\mathrm{Pic}(X)$ is generated by $\mathcal{O}_X(1)$, as required. $\square$

Chapter 2

Cohomology

2.1 Review of homological algebra

In this section, we present briefly some concepts of homological algebra which will be used in the cohomology of sheaves. From now, $\mathscr{A}$ and $\mathscr{B}$ denote two abelian categories. The concepts presented can be found in [Har77] and [Rot08].

Definition 2.1.1 (Complex)
A complex, *or* cochain complex, *$(A^\bullet, d)$ in $\mathscr{A}$ is a collection of objects $A^n \in \mathscr{A}$, $n \in \mathbb{Z}$ and morphisms $d^n : A^n \longrightarrow A^{n+1}$ such that $d^{n+1}d^n = 0$.*

Examples 2.1.2 (i) All exact sequences are complexes.

(ii) In differential geometry the algebras of n-forms $\Omega^n(M)$ of a manifold M and the exterior derivatives $d^n : \Omega^n(M) \longrightarrow \Omega^{n+1}(M)$ form a complex. Its cohomology yields to the de Rham cohomology.

Definition 2.1.3 (Morphism of complexes)
Let $(A^\bullet, d)$ and $(B^\bullet, d)$ be two complexes in $\mathscr{A}$. A morphism f from $A^\bullet$ to $B^\bullet$, denoted by $f : A^\bullet \longrightarrow B^\bullet$, is a collection of morphisms $f^n : A^n \longrightarrow B^n$ such that $f^{n+1}d^n = d^n f^n$, for all $n \in \mathbb{Z}$.

If $f : A^\bullet \longrightarrow B^\bullet$ and $g : A^\bullet \longrightarrow C^\bullet$ are two morphisms of complexes, then we have a morphism of complexes $gf : A^\bullet \longrightarrow C^\bullet$ defined by $(gf)^n = g^n f^n$. Hence, we have the category $\mathbf{Comp}(\mathscr{A})$ of complexes in $\mathscr{A}$. One can check that $\mathbf{Comp}(\mathscr{A})$ is an abelian category. Furthermore, the kernel and the image of a morphism $f : A^\bullet \longrightarrow B^\bullet$ is taken componentwise, that is: the kernel of f is $\big((\ker f)^\bullet, n\big)$ where $(\ker f)^n = \ker f^n$ and i_n is the "inclusion" of $\ker f^n$ into A^n.

Definition 2.1.4 (Cohomology object)
Let $A^\bullet$ be a complex. The nth cohomology object*, denoted by $h^n(A^\bullet)$, is defined as* $\ker d^n / \operatorname{im} d^{n+1}$.

Remark 2.1.5

To construct this nth cohomology object, consider the following situation:

$$\begin{tikzcd}
A^{n-1} \arrow[rr, "d^{n-1}"] \arrow[dr] \arrow[ddrr, dashed, bend right, "\gamma^{n-1}"'] & & A^n \arrow[r, "d^n"] & A^{n+1} \\
& \operatorname{im} d^{n-1} \arrow[ur, hook] \arrow[dr, dashed, "\delta^{n-1}"] & & \\
& & \ker d^n \arrow[uu, hook] &
\end{tikzcd}$$

where γ^{n-1} is induced by the universal property of the kernel and δ^{n-1} is induced by the universal property of the image (since any equalizer is a monomorphism). Then take $h^n(A^\bullet) = \operatorname{coker} \delta^{n-1}$.

We recall the following theorem.

Theorem 2.1.6 (Mitchell's embedding theorem)
Every small abelian category admits a full, faithful and exact functor to the category of left R-modules for some ring R.

This theorem allows us to make proofs by diagram-chasing.

Proposition 2.1.7
Let $f : A^\bullet \longrightarrow B^\bullet$. Then f induces morphisms $h^n(f) : h^n(A^\bullet) \longrightarrow h^n(B^\bullet)$.

Proposition 2.1.8
Let $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ be a short exact sequence of complexes in $\mathscr{A}$. Then there exists morphisms $\delta^i : h^i(C^\bullet) \longrightarrow h^{i+1}(A^\bullet)$ giving rise to a long exact sequence

$$\ldots \longrightarrow h^i(A^\bullet) \longrightarrow h^i(B^\bullet) \longrightarrow h^i(C^\bullet) \longrightarrow h^{i+1}(A^\bullet) \longrightarrow \ldots$$

Proof. Consider the following diagram:

$$\begin{tikzcd}
& A^n/\operatorname{im} d_A^{n-1} \arrow[r, "\overline{f^n}"] \arrow[d, "\overline{d_A^n}"] & A^n/\operatorname{im} d_B^{n-1} \arrow[r, "\overline{g^n}"] \arrow[d, "\overline{d_B^n}"] & A^n/\operatorname{im} d_C^{n-1} \arrow[r] \arrow[d, "\overline{d_C^n}"] & 0 \\
0 \arrow[r] & \ker d_A^{n+1} \arrow[r, "f^{n+1}|_{\ker d_A^{n+1}}"] & \ker d_B^{i+1} \arrow[r, "g^{n+1}|_{\ker d_B^{n+1}}"] & \ker d_C^{i+1} &
\end{tikzcd}$$

where:

- $\overline{d_A^n} : a + \operatorname{im} d_A^{n-1} \longmapsto d_A^n(a)$;
- $\overline{f^n} : a + \operatorname{im} d_A^{n-1} \longmapsto f(a) + \operatorname{im} d_B^{n-1}$;
- $\overline{d_B^n}, \overline{d_C^n}$ and $\overline{g^n}$ are defined in a similar way.

Then one can check that the two rows are exact and the following equalities hold:

$$\begin{array}{lll}
\ker \overline{d_A^n} = h^n(A), & \ker \overline{d_B^n} = h^n(B), & \ker \overline{d_C^n} = h^n(C), \\
\operatorname{coker} \overline{d_A^n} = h^{n+1}(A), & \operatorname{coker} \overline{d_B^n} = h^{n+1}(B), & \operatorname{coker} \overline{d_C^n} = h^{n+1}(C).
\end{array}$$

Finally, the snake lemma gives the result. $\square$

Definition 2.1.9 (Injective object)
An object I of $\mathscr{A}$ is injective *if the functor* $\mathrm{Hom}_{\mathscr{A}}(-, I)$ *is exact.*

Remarks & examples 2.1.10 (i) If $\mathscr{A}$ is the category of R-modules, then this definition coincide with the definition of an injective module.

(ii) A product of injective objects is injective.

(iii) The object 0 is injective.

Definition 2.1.11 (Injective resolution)
Let A be an object of $\mathscr{A}$. An injective resolution *is an exact sequence in $\mathscr{A}$*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

such that each I^n is injective.

Definition 2.1.12 (Deleted injective resolution)
Let A be an object of $\mathscr{A}$ and $I = 0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$ be an injective resolution for A. Then the deleted injective resolution I^A *of A is the sequence*

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \dots$$

Definition 2.1.13
If every object of $\mathscr{A}$ is isomorphic to a subobject of an injective object, we say that $\mathscr{A}$ has enough injectives.

Proposition 2.1.14
If $\mathscr{A}$ has enough injectives, then every object of A has an injective resolution.

Proof. Let A be an object of $\mathscr{A}$. By hypothesis, there exists an injective object I^0 of $\mathscr{A}$ and a monomorphism $\varepsilon : A \longrightarrow I^0$. Now, suppose that we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} I^n.$$

Consider the monomorphism $\alpha : I^n/_{\mathrm{im}\, d^{n-1}} \longrightarrow I^{n+1}$, for some injective object I^{n+1}. Then take $d^n = \alpha\pi$, where π is the canonical map from I^n to $I^n/_{\mathrm{im}\, d^{n-1}}$. $\square$

From now, we suppose that $\mathscr{A}$ has enough injectives.

Definition 2.1.15 (Right derived functor)
Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a covariant additive left exact functor. Then we construct the right derived functors R^nF *as follows:*

- *For each object A of $\mathscr{A}$, fix an injective resolution $I_A^\bullet$ of A.*
- *For all $n \in \mathbb{N}_0$ and every $A \in \mathscr{A}$, set $R^nF(A) = h^n\big(FI_A^A\big)$, where I_A^A is the deleted resolution of A.*

Remark 2.1.16
The sequence $0 \longrightarrow FI^0 \xrightarrow{Fd^1} FI^1 \longrightarrow \dots$ may fails to be exact but is a complex.

Remark 2.1.17
Let F and A as in definition. Then $R^0F = \ker Fd^0 \cong F(A)$ since F is left exact.

Theorem 2.1.18
Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a covariant additive left exact functor. Then $R^nT : \mathscr{A} \longrightarrow \mathscr{B}$ is an additive covariant functor for every $n \in \mathbb{N}_0$.

Proof. See the Comparaison Theorem of [Rot08]. □

Theorem 2.1.19
Let $F : \mathscr{A} \longrightarrow \mathscr{B}$ be a covariant additive left exact functor. Then:

- (i) *The definition of R is independant of the choice of the injective resolutions.*
- (ii) *For each short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ in $\mathscr{A}$ and for each $n \in \mathbb{N}_0$, there is a morphism $\delta^n : R^nF(C) \longrightarrow R^{n+1}F(A)$ such that we obtain a long exact sequence*

$$\ldots \longrightarrow R^nF(A) \longrightarrow R^nF(B) \longrightarrow R^nF(C) \xrightarrow{\delta^n} R^{n+1}F(A) \longrightarrow \ldots$$

Proof. See Proposition 6.20, Corollary 6.22 and Theorem 6.27 of [Rot08]. □

Corollary 2.1.20
Let I be an injective object. Then $R^nFI = 0$ for all $n \in \mathbb{N}$.

Proof. Since the choice of the injective resolution is irrelevant, we can choose the following one:

$$0 \longrightarrow I \xrightarrow{\mathrm{id}} I \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

Therefore, $R^nFI = 0$ for all $n \in \mathbb{N}$. □

2.2 Cohomology of sheaves

2.2.1 Derived functor definition

We know that $\mathfrak{Ab}(X)$ (the category of sheaves on a topological space) and $\mathscr{M}od(\mathcal{O}_X)$ (the category of $\mathcal{O}_X$-modules) are abelian categories. Furthermore, since the global section functor $\Gamma(X, -)$ which sends a sheaf $\mathscr{F}$ to $\Gamma(X, \mathscr{F}) = \mathscr{F}(X)$ is an additive left exact functor, we can consider its right derived functors. It can easily be shown (Proposition II.2.2 of [Har77]) that $\mathscr{M}od(\mathcal{O}_X)$ has enough injectives. In particular, $\mathfrak{Ab}(X)$ has enough injectives (a sheaf can be viewed as a $\mathbb{Z}$-module). Hence we can give the following definition:

Definition 2.2.1
For any topological space X, we define the cohomology functors *$H^n(X, -)$ as the right derived functors of* $\Gamma(X, -)$.

Remark 2.2.2
In this definition, we consider just the abelian group sheaf structure of a sheaf $\mathscr{F}$, even if $\mathscr{F}$ has the extra structure of an $\mathcal{O}_X$-module. However, it can be shown that if $(X, \mathcal{O}_X)$ is a ringed space, the derived functors of $\Gamma(X, -)$ from $\mathscr{M}od(\mathcal{O}_X)$ to **Ab** coincide with the $H^n(X, -)$ as defined above.

2.2.2 Čech cohomology

Let X be a topological space, $\mathscr{U} = \{U_i\}_{i\in I}$ be a covering of X and consider a well-order $<$ on I. For $i_0, \dots, i_n \in I$, we denote by $U_{i_0,\dots,i_n}$ the set $U_{i_0} \cap \dots \cap U_{i_n}$. Now, let $\mathscr{F}$ be a sheaf on X and $n \in \mathbb{N}_0$. We define the following group:

$$C^n(\mathscr{U}, \mathscr{F}) = \prod_{i_0<\dots<i_n} \mathscr{F}(U_{i_0,\dots,i_n}).$$

Then, we define the map

$$\begin{aligned} d^n : C^n(\mathscr{U}, \mathscr{F}) &\longrightarrow C^{n+1}(\mathscr{U}, \mathscr{F}) \\ s &\longmapsto d^n s, \quad (d^n s)_{i_0,\dots,i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j s_{i_0,\dots,\widehat{i_j},\dots i_{n+1}}\big|_{U_{i_0,\dots,i_{n+1}}}, \end{aligned}$$

where $i_0, \dots, \widehat{i_j}, \dots i_{n+1}$ means $i_0, \dots, i_{j-1}, i_{j+1}, \dots, i_{n+1}$. One can check that the composition $d^{n+1}d^n = 0$.

Remark 2.2.3
Although it will not be mentioned explicitely, in these constructions, the $i_0, \dots, i_n$ are supposed distinct.

Definition 2.2.4 (Čech cohomology group)
Let X be a topological space, $\mathscr{F}$ be a sheaf on X and $\mathscr{U}$ as above and $n \in \mathbb{N}_0$. Then the nth Čech cohomology group *of $\mathscr{F}$ (with respect to $\mathscr{U}$), is $h^n C^\bullet(\mathscr{U}, \mathscr{F})$ and is denoted $\check{H}^n(\mathscr{U}, \mathscr{F})$.*

Proposition 2.2.5 (Functoriality of $\check{H}$)
Let X and $\mathscr{U}$ as in the definition and $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ a morphism of sheaves on X. Then α induces a morphism of complexes $\alpha : C^\bullet(\mathscr{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathscr{U}, \mathscr{G})$. Furthermore, this association is functorial. Therefore, α induces a homomorphism between $\check{H}^n(\mathscr{U}, \mathscr{F})$ and $\check{H}^n(\mathscr{U}, \mathscr{G})$ for all $n \in \mathbb{N}_0$.

Proof. First define the following homomorphism:

$$\begin{aligned} \alpha^n : C^n(\mathscr{U}, \mathscr{F}) &\longrightarrow C^n(\mathscr{U}, \mathscr{G}) \\ s &\longmapsto \alpha^n(s), \quad \big(\alpha^n(s)\big)_{i_0,\dots,i_n} = \alpha_{U_{i_0,\dots,i_n}}(s_{i_0,\dots,i_n}). \end{aligned}$$

Then it is easy to see that this homomorphism commutes with d. □

Let $\{\mathscr{F}_i\}_{i\in I}$ be a collection of sheaves on a topological space X. Since the sheaf product is defined as $\left(\prod \mathscr{F}_i\right)(U) = \prod \mathscr{F}_i(U)$ for all open set U and since restriction maps on the sheaf product are defined componentwise, we have the following result.

Proposition 2.2.6
Direct product commutes with Čech cohomology.

Theorem 2.2.7
Let X be a noetherian separated scheme, $\mathscr{U}$ be an open affine cover of X and $\mathscr{F}$ a quasi-coherent sheaf on X. Then for all $n \in \mathbb{N}_0$ we have $\check{H}^n(\mathscr{U}, \mathscr{F}) \cong H^n(X, \mathscr{F})$.

Proof. See [Har77, II.4.5]. □

This theorem is very useful to compute cohmology groups since it is not easy to find an injective resolution of a sheaf.

2.2.2.1 Remarks, examples and first properties

Remarks 2.2.8
We make the following remarks:

(i) We have $\check{H}^0(\mathscr{U},\mathscr{F}) \cong \mathscr{F}(X)$. To see that, take $\alpha \in \check{H}^0(\mathscr{U},\mathscr{F})$. If $i,j \in I$ are distinct elements, then $i<j$ or $j<i$ and the condition $\alpha \in \ker d^0$ implies that $\alpha_i\big|_{U_i\cap U_j} = \alpha_j\big|_{U_i\cap U_j}$. So the α_i glue to an element $t \in \mathscr{F}(X)$.

(ii) If $|I| = N < \infty$, then $\check{H}^n(\mathscr{U},\mathscr{F}) = 0$ for all $n \geq N$. Indeed, we cannot fnd more than N distinct elements in I.

(iii) Although $\check{H}^0(\mathscr{U},\mathscr{F}) \cong R^0\Gamma(\mathscr{F})$, this is not the case for all n. In particular, $\check{H}$ may fail to give a long exact sequence. Consider $X = \mathbb{C}^*$, $\mathscr{U} = \{X\}$ (which means that $\check{H}^n(\mathscr{U},\mathscr{F}) = 0$ for all $n \geq 1$) and the following exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\Phi} \mathcal{O}^* \longrightarrow 0,$$

where $\mathbb{Z}$ is the constant sheaf ($\mathbb{Z}(U)$ is the set of all continuous functions from U to $\mathbb{Z}$, endowed with the discrete topology), $\mathcal{O}$ is the sheaf of holomorphic functions of X, $\mathcal{O}^*$ is the sheaf of non-vanishing homolomorphic functions and Φ maps f to $\exp(2\pi f)$. Since $\check{H}^0(\mathscr{U},\mathscr{F}) = \mathscr{F}(X)$ for every sheaf $\mathscr{F}$, we have

$$0 \longrightarrow \mathbb{Z}(X) \longrightarrow \mathcal{O}(X) \xrightarrow{\Phi_X} \mathcal{O}^*(X).$$

And this sequences is not exact if we add $\check{H}^1(\mathscr{U},\mathbb{Z}) = 0$ on the right, since id is not in the image of Φ_X.

Example 2.2.9
Let $X = S^1$, the unit circle and let $\mathscr{U} = \{U_0, U_1\}$ where U_0 and U_1 are two open half-circles which overlap. We consider the constant sheaf $\mathbb{Z}$ on X. Let us show that $\check{H}^0(\mathscr{U},\mathbb{Z}) = \mathbb{Z}$, $\check{H}^1(\mathscr{U},\mathbb{Z}) = \mathbb{Z}$ and $\check{H}^2(\mathscr{U},\mathbb{Z}) = 0$. We have

$$C^0(\mathscr{U},\mathbb{Z}) = \Gamma(U_0,\mathbb{Z}) \times \Gamma(U_1,\mathbb{Z}), \quad C^1(\mathscr{U},\mathbb{Z}) = \mathbb{Z}(U_0 \cap U_1) = \mathbb{Z}\times\mathbb{Z}.$$

The first equality holds because U_0 and U_1 are connected (recall that $\mathbb{Z}$ is endowed with the discrete topology) and the second one because there is two connected component. We have

$$\begin{tikzcd} \mathbb{Z}(U_0)\times\mathbb{Z}(U_1) \arrow[r, "d^0"] \arrow[d, "\varphi"'] & \mathbb{Z}(U_0\cap U_1) \arrow[d, "\psi"] \\ \mathbb{Z}\times\mathbb{Z} \arrow[r, "\widetilde{d^0}"'] & \mathbb{Z}\times\mathbb{Z} \end{tikzcd}$$

where $\varphi(f,g) = \big(f(-1,0), g(1,0)\big)$ and $\psi(h) = \big(h(-1,0), h(1,0)\big)$. Then we have $\widetilde{f^0}(a,b) = (b-a, b-a)$. Since $\check{H}^2(\mathscr{U},\mathbb{Z}) = 0$, we have $\check{H}^1(\mathscr{U},\mathbb{Z}) = \mathbb{Z}$ (we knew already that $\check{H}^0(\mathscr{U},\mathbb{Z}) = \mathbb{Z}$).

Example 2.2.10
Let k be a field and consider $\mathbb{A}^n_k$ for some $n \in \mathbb{N}$. Then $H^m(\mathbb{A}^n_k,\mathscr{F}) = 0$ for all $m \in \mathbb{N}$ and every quasi-coherent sheaf $\mathscr{F}$ on $\mathbb{A}^n_k$. Indeed, since $\{\mathbb{A}^n_k\}$ is itself an affine open covering of the space, the result follows from Theorem 2.2.7 and Remark 2.2.8.

Example 2.2.11
Let k be a field and consider the subvariety $U = D(x) \cup D(y)$ of $X = \operatorname{Spec} k[x, y]$. Let $\mathscr{U}$ be the affine covering $\{D(x), D(y)\}$ of X. We will show that $H^1(\mathscr{U}, \mathcal{O}_U)$ is an infinite vector space over k. By Theorem 2.2.7 it is sufficient to show that $\check{H}^1(\mathscr{U}, \mathcal{O}_X)$ is an infinite vector space over k. Since $\check{H}^2(\mathscr{U}, \mathcal{O}_U) = 0$, we know that we have the equality $\check{H}^1(\mathscr{U}, \mathcal{O}_U) = C^1(\mathscr{U}, \mathcal{O}_U)/_{\operatorname{im} d^0}$. Furthermore, we have

$$
\begin{aligned}
C^0(\mathscr{U}, \mathcal{O}_U) &\cong k[x, y]_x \times k[x, y]_y = k\left[x, y, \frac{1}{x}\right] \times k\left[x, y, \frac{1}{y}\right] \\
C^1(\mathscr{U}, \mathcal{O}_U) &\cong k[x, y]_{xy} = k\left[x, y, \frac{1}{y}, \frac{1}{x}\right].
\end{aligned}
$$

The map $d^0 : k\left[x, y, \frac{1}{x}\right] \times k\left[x, y, \frac{1}{y}\right] \longrightarrow k\left[x, y, \frac{1}{y}, \frac{1}{x}\right]$ is the following:

$$(f, g) \longmapsto g - f.$$

Consider the vector space V with basis $\left\{x^i y^j : i, j \in \mathbb{Z}, i, j < 0\right\}$. We want to show that $\check{H}^1(\mathscr{U}, \mathcal{O}_U) \cong V$. We define a map $\phi : k\left[x, y, \frac{1}{y}, \frac{1}{x}\right] \longrightarrow V$ by extending by linearity the map

$$x^i y^j \longmapsto \begin{cases} x^i y^j & \text{if } i < 0, j < 0, \\ 0 & \text{else.} \end{cases}$$

Since $\ker \phi = \operatorname{im} d^0$, the result follows.

Proposition 2.2.12
Let X and Y be noetherian separated schemes and $f : X \longrightarrow Y$ be an affine morphism. Then $H^n(X, \mathscr{F}) \cong H^n(Y, f_* X)$ for all $n \in \mathbb{N}_0$ and every quasi-coherent sheaf $\mathscr{F}$ on X.

Proof. Let $\mathscr{F}$ be a quasi-coherent sheaf. From assumption $f_*\mathscr{F}$ is also quasi-coherent and Theorem 2.2.7 we need to show that $\check{H}^n(\mathscr{U}, \mathscr{F}) \cong \check{H}^n(\mathscr{U}', f_*\mathscr{F})$ for some affine coverings $\mathscr{U}$ of X and $\mathscr{U}'$ of Y. Let $\mathscr{U}' = \{V_i\}_{i \in I}$ be an affine open covering of Y. By assumption, $\mathscr{U} = \left\{U_i = f^{-1}V_i : i \in I\right\}$ is an affine open covering of X. Then we see that $C^n(\mathscr{U}, \mathscr{F}) = C^n(\mathscr{U}', f_*\mathscr{F})$. $\square$

Proposition 2.2.13
Let X a noetherian topological space. Let Λ be a directed set and $\left\{\mathscr{F}_\lambda\right\}_{\lambda \in \Lambda}$ be a direct system of sheaves on X. Then the association $U \longmapsto L(U) := \varinjlim \mathscr{F}_\lambda(U)$ defines a sheaf on X.

Proof. It is clear that L is a presheaf of X. Let U be an open set of X and $\{U_i\}_{i \in I}$ be an open cover of U. Since X is noetherian, the subspace U is compact and so $U = U_1 \cup \ldots \cup U_n$. Let $[\lambda, s] \in L(U)$ such that $[\lambda, s]\big|_{U_i} = \left[\lambda, s\big|_{U_i}\right] = 0$. Thus, there exist $\lambda_1, \ldots, \lambda_n$ such that $\rho_{U_i}^{\lambda, \lambda_i}\left(s\big|_{U_i}\right) = 0$. Then choose $\mu \in \Lambda$ such that $\lambda_i \leq \mu$ for all $1 \leq i \leq n$. We have

$$\rho_U^{\lambda, \mu}(s)\big|_{U_i} = \rho_{U_i}^{\lambda, \mu}\left(s\big|_{U_i}\right) = \rho_{U_i}^{\lambda, \mu} \rho_{U_i}^{\lambda, \lambda_i}\left(s\big|_{U_i}\right) = 0.$$

Therefore, $\rho_U^{\lambda, \mu}(s) = 0$ which implies that $[\lambda, s] = 0$, as required. The glueing condition can be checked in a similar way. $\square$

The following proposition is exercise 5.2.6 of [Liu06].

Proposition 2.2.14
Let X a noetherian topological space and $\mathscr{U}$ be an open cover of X. Let Λ be a directed set and $\{\mathscr{F}_\lambda\}_{\lambda\in\Lambda}$ be a direct system of sheaves on X. Then, for all $n \in \mathbb{N}_0$, we have

$$\varinjlim \check{H}^n(\mathscr{U}, \mathscr{F}_\lambda) \cong \check{H}^n\left(\mathscr{U}, \varinjlim \mathscr{F}_\lambda\right).$$

Proof. We denote by $\mathscr{F}$ the sheaf $\varinjlim \mathscr{F}_\lambda$. First remark that since $\check{H}^n$ is a functor, $H^n\left(\mathscr{U}, \varinjlim \mathscr{F}_\lambda\right)$ is a direct system of groups and the question makes sense. For this proof, we denote by I a multi-index $i_0, \dots, i_n$, where $i_0 < \dots < i_n$. Then, if s is an element of $C^n(\mathscr{U}, \mathscr{F})$, we can write s_I and U_I with the same meaning as above. Furthermore, if $0 \leq j \leq n$, then $I, \hat{j}$ will denote $i_0, \dots, \hat{j}, \dots, i_n$. Now, fix $n \in \mathbb{N}_0$ and for every $\alpha \in \Lambda$ consider

$$\begin{aligned}\tilde{\varphi}_\alpha : C^n(\mathscr{U}, \mathscr{F}_\lambda) &\longrightarrow C^n(\mathscr{U}, \varinjlim \mathscr{F}_\lambda)\\ s &\longmapsto \tilde{\varphi}_\alpha(s), \quad \tilde{\varphi}_\alpha(s)_I = [\lambda, s_I].\end{aligned}$$

We first check that $\operatorname{im} \tilde{\varphi}_\alpha\big|_{\ker d_\alpha^n} \subset \ker d^n$:

$$\begin{aligned}\left(d^n \tilde{\varphi}_\alpha(s)\right)_J &= \sum_{j=0}^{n+1} (-1)^j \tilde{\varphi}_\alpha(s)_{J,\hat{j}}\big|_{U_J} = \sum_{j=0}^{n+1} (-1)^j \left[\alpha, s_{J,\hat{j}}\right]\big|_{U_J}\\ &= \left[\alpha, \sum_{j=0}^{n+1} (-1)^j s_{J,\hat{j}}\big|_{U_J}\right] = \left[\alpha, d_\alpha^n(s)_J\right] = 0.\end{aligned}$$

A similar calculation shows that if $s \in \operatorname{im} d_\alpha^{n-1}$, then $\tilde{\varphi}_\alpha(s) \in \operatorname{im} d^{n-1}$. So we obtain maps $\varphi_\lambda : H^n(\mathscr{U}, \mathscr{F}_\alpha) \cong \check{H}^n\left(\mathscr{U}, \varinjlim \mathscr{F}_\lambda\right)$. Since these maps are compatible with the morphisms $\rho_{\alpha,\beta}$ we get a morphism from $\varinjlim \check{H}^n(\mathscr{U}, \mathscr{F}_\lambda)$ to $\check{H}^n\left(\mathscr{U}, \varinjlim \mathscr{F}_\lambda\right)$. Then one can check that this morphism is an isomorphism. □

2.2.2.2 Projective spaces and twisted sheafs

We consider in this subsection the projective space and some twisted sheafs. Let's take some algebraically closed field k, $n \in \mathbb{N}_0$ and set $X_n = \operatorname{Proj} k[x_0, \dots, x_n]$. We have seen that the scheme X_n is separated and noetherian and we can take the affine open covering $\mathscr{U} = \{D_+(x_0), \dots, D_+(x_n)\}$. We compute here a few Čech cohomology groups for the coherent sheaves $\mathcal{O}_X(m)$ on X_n with respect to this covering. Note that by Theorem 2.2.7 this coincides with the groups $H^r\big(X_n, \mathcal{O}_X(m)\big)$.

Example 2.2.15 $(n = 0)$
We have $\operatorname{Proj} k[x]$ equals $\operatorname{Spec} k$ which implies that $H^1\big(X_0, \mathcal{O}_X(m)\big) = 0$ for every m.

Example 2.2.16 $(n = 1, m = 0)$
We take the open covering $\mathscr{U} = \{D_+(x), D_+(y)\}$ of $X = \mathbb{P}^1$ and set $R = k[x, y]$. We will show that $\check{H}^0(\mathbb{P}^1, \mathcal{O}_X) = k$ and $\check{H}^1(\mathbb{P}^1, \mathcal{O}_X) = 0$. We know that $\check{H}^0(\mathscr{U}, \mathcal{O}_X)$ is the global sections of $\mathcal{O}_X$, that is the polynomials of degree 0 of $k[x, y]$. We have also the following equalities

$$\begin{aligned}C^0(\mathscr{U}, \mathcal{O}_X) &= \mathcal{O}_X\big(D_+(x)\big) \times \mathcal{O}_X\big(D_+(y)\big) = k[x, y]_{(x)} \times k[x, y]_{(y)}\\ C^1(\mathscr{U}, \mathcal{O}_X) &= \mathcal{O}_X\big(D_+(xy)\big) = k[x, y]_{(xy)}.\end{aligned}$$

Recalling that $R_{(f)}$ consists of elements of degree 0 in R_f, we find:

$$C^0(\mathscr{U}, \mathcal{O}_X) = k\left[\frac{y}{x}\right] \times k\left[\frac{x}{y}\right], \quad C^1(\mathscr{U}, \mathcal{O}_X) = k\left[\frac{x}{y}, \frac{y}{x}\right].$$

Furthermore, the map d^0 send a pair of polynomials (f, g) to the polynomial $g - f$. Since $C^2(\mathscr{U}, \mathcal{O}_X) = 0$, we know that $k\left[\frac{x}{y}, \frac{y}{x}\right] / \operatorname{im} d^0$. Now, it is easy to see that k, $\frac{x}{y}$ and $\frac{y}{x}$ are all included in $\operatorname{im} d^0$. Hence, $\check{H}^1(\mathscr{U}, \mathcal{O}_X) = 0$.

More general calculations We now consider the case where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. We have the open covering $\mathscr{U} = \{D_+(x_0), \ldots, D_+(x_n)\}$. If we fix $i_0 < \ldots < i_k$ for some $k \in \mathbb{N}$ and set $R = k[x_0, \ldots, x_n]$, we have

$$\begin{aligned}
\mathcal{O}_X(m)\big(D_+(x_{i_0}) \cap \ldots \cap D_+(x_{i_k})\big) &= \mathcal{O}_X(m)\big(D_+(x_{i_0} \cdot \ldots \cdot x_{i_k})\big) \\
&= \widetilde{R(m)}\big(D_+(x_{i_0} \cdot \ldots \cdot x_{i_k})\big) \\
&\overset{1.1.57}{=} \Gamma\Big(\operatorname{Spec} R_{(x_{i_0} \cdot \ldots \cdot x_{i_k})}, \big(R(m)_{(x_{i_0} \cdot \ldots \cdot x_{i_k})}\big)^{\sim}\Big).
\end{aligned}$$

Looking at the proof of Proposition $II.2.5$ of [Har77], we find that the last term consist of homogeneous elements of degree m in $k[x_0, \ldots, x_n]_{x_{i_0} \cdot \ldots \cdot x_{i_k}}$ (with the standard graduation), that is: homogeneous elements of degree m in $k\left[x_0, \ldots, x_n, \frac{1}{x_{i_0}}, \ldots, \frac{1}{x_{i_k}}\right]$. We will denote this vector space by $k\left[x_0, \ldots, x_n, \frac{1}{x_{i_0}}, \ldots, \frac{1}{x_{i_k}}\right]_m$.

Proposition 2.2.17
Let $n \in \mathbb{N}$ and $X = \operatorname{Proj} k[x_0, \ldots, x_n]$. Then we have

(i) $k[x_0, \ldots, x_n] \cong \bigoplus_{m \in \mathbb{Z}} H^0\big(X, \mathcal{O}_X(m)\big)$;

(ii) $H^n\big(X, \mathcal{O}_X(-n-1)\big) \cong k$.

Proof. (i) Follows directly from Proposition 1.1.61 and Remark 2.2.8.

(ii) We set $m = -n-1$. First, remark that since $|\mathscr{U}| = n+1$, we have $\ker d^n = C^n\big(\mathscr{U}, \mathcal{O}_X(m)\big)$. By the above calculations, we know that $C^n\big(\mathscr{U}, \mathcal{O}_X(m)\big)$ is equal to $k\left[x_0, \ldots, x_n, \frac{1}{x_0}, \ldots, \frac{1}{x_n}\right]_m$. On the other hand, we have

$$C^{n-1}\big(\mathscr{U}, \mathcal{O}_X(m)\big) = \prod_{j=0}^{n+1} k\left[x_0, \ldots, x_n, \frac{1}{x_0}, \ldots, \widehat{\frac{1}{x_j}}, \ldots \frac{1}{x_n}\right]_m$$

and the map d^{n-1} from $C^{n-1}\big(\mathscr{U}, \mathcal{O}_X(m)\big)$ to $C^n\big(\mathscr{U}, \mathcal{O}_X(m)\big)$ is just the "alternating inclusion":

$$d^{n-1}(f_0, \ldots, f_n) = f_0 - f_1 + \ldots + (-1)^n f_n.$$

As a k-vector space $C^n\big(\mathscr{U}, \mathcal{O}_X(m)\big)$ admits the following basis:

$$\big\{x_0^{a_0} \cdot \ldots \cdot x_n^{a_n} : a_i \in \mathbb{Z}, a_0 + \ldots + a_n = m\big\}.$$

Furthermore, the vector space $\operatorname{im} d^{n-1} \subset C^n\big(\mathscr{U}, \mathcal{O}_X(m)\big)$ has the basis

$$\big\{x_0^{a_0} \cdot \ldots \cdot x_n^{a_n} : a_i \in \mathbb{Z}, a_0 + \ldots + a_n = m, \exists j \text{ such that } a_j \geq 0\big\}.$$

Hence, $H^n\big(X, \mathcal{O}_X(-n-1)\big)$ has $\big\{x_0^{a_0} \cdot \ldots \cdot x_n^{a_n} : a_i \in \mathbb{Z}, a_0 + \ldots + a_n = m, a_i < 0\big\}$ as a basis. The only possibility to have such monomials have degree $-n-1$ is $a_0 = a_1 = \ldots = a_n = -1$. Therefore, $H^n\big(X, \mathcal{O}_X(-n-1)\big)$ is a k-vector space of dimension 1.

□

2.2.3 More on Čech cohomology

2.2.3.1 An alternative definition

The aim of this section is to present an equivalent definition of cohomology as the limit over all open coverings of the Čech cohomology groups.

Definition 2.2.18 (Alternating n-cochain)
Let X be a topological space and $\{U_i\}_{i\in I}$ an open covering of X. A n-cochain f in $\prod_{(i_0,\dots,i_n)\in I^{n+1}} \mathscr{F}(U_i)$ is alternating *if the two following conditions hold:*

(*i*) *For every permutation $\sigma \in S_{n+1}$, we have $f_{i_0,\dots,i_n} = \operatorname{sgn}(\sigma) f_{\sigma(i_0),\dots,\sigma(i_n)}$.*

(*ii*) *If there exist $0 \leq k < l \leq n$ such that $i_k = i_l$, then $f_{i_0,\dots,i_n} = 0$.*

Notation 2.2.19 (Group of alternating cochains)
The group of alternating n-cochains is denoted by $C'^n(\mathscr{U}, \mathscr{F})$.

Remark 2.2.20
We can define the coboundary maps as above:

$$\begin{aligned} d^n : C'^n(\mathscr{U}, \mathscr{F}) &\longrightarrow C'^{n+1}(\mathscr{U}, \mathscr{F}) \\ s &\longmapsto d^s f, (d^n s)_{i_0,\dots,i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j s_{i_0,\dots,\widehat{i_j},\dots i_{n+1}}\big|_{U_{i_0,\dots,i_{n+1}}}. \end{aligned}$$

Then, one can check that an alternating is mapped to an alternating cochain and that $d^{n+1}d^n = 0$.

Proposition 2.2.21
The two definitions $\left(C^\bullet, d^\bullet\right)$ and $\left(C'^\bullet, d^\bullet\right)$ give rise to the same groups of cohomology (up to isomorphism).

Proof. See Proposition 5.2.3 and Corollary 5.2.4 of [Liu06]. □

2.2.3.2 Refinement of open coverings

Now we want to relate Čech cohomology when we have different coverings. To do this we need to introduce some more notions.

Definition 2.2.22 (Refinement of an open covering)
Let $\mathscr{U} = \{U_i\}_{i\in I}$ and $\mathscr{U}' = \{V_j\}_{j\in J}$ be two open covering of X. We say that $\mathscr{U}'$ is a refinement *of $\mathscr{U}$ if there exists a map $\lambda : J \longrightarrow I$ such that $V_j \subset U_{\lambda(j)}$ for all $j \in J$.*

Proposition 2.2.23
Let $\mathscr{U} = \{U_i\}_{i\in I}$ be an open covering of X and $\left(\mathscr{U}' = \{V_j\}_{j\in J}, \lambda\right)$ a refinement of $\mathscr{U}$. This refinement gives rise to a morphism $\tau^n : \check{H}^n(\mathscr{U}, \mathscr{F}) \longrightarrow \check{H}^n(\mathscr{U}', \mathscr{F})$ for every $n \in \mathbb{N}_0$.

Proof. For every $n \in \mathbb{N}_0$, we define

$$\begin{aligned} \lambda^n : C^n(\mathscr{U}, \mathscr{F}) &\longrightarrow C^n(\mathscr{U}', \mathscr{F}) \\ f &\longmapsto \lambda^n f, \left(\lambda^n f\right)_{j_0,\dots,j_n} = f_{\lambda(j_0),\dots,\lambda(j_n)}\big|_{V_{j_0,\dots,j_n}}. \end{aligned}$$

Then, it is easy to see that these maps commute with the coboundary maps. So we get a morphism $\tau^n : \check{H}^n(\mathscr{U}, \mathscr{F}) \longrightarrow \check{H}^n(\mathscr{U}', \mathscr{F})$. □

Proposition 2.2.24
The map we get in the previous Proposition does not depend on the choice of λ.

Proof. See §21 ("Passage d'un recouvrement à un recouvrement plus fin") of [Ser55]. □

We want to use this "refinement relation" to consider the direct limit on the class of open covering. There are two problems: the class of open coverings is not a set and the relation would like to define is not antisymmetric. We restrict ourselves to the open coverings which contain each open set at most one time. The class of such open coverings is a set and we can define the following relation on it:

$$\mathscr{U} = \{U_i\}_{i\in I} \leq \mathscr{U}' = \{V_j\}_{j\in J} \Leftrightarrow \mathscr{U}' \text{ is a refinement of } \mathscr{U}.$$

We solve the second problem as follows:

Proposition 2.2.25
Let Y *bet a set and let* $\leq$ *be a binary reflexive and transitive relation on* Y. *Define a equivalence on* Y *as follows:*

$$a \sim b \Leftrightarrow a \leq b \text{ and } b \leq a.$$

Then $\leq$ *gives rise to a partial order on* $Y/_{\sim}$.

Proof. It is clear that $\sim$ is an equivalence relation on Y. Furthermore, $[a] \leq [b]$ if and only if $a \leq b$ is a well defined partial order on $Y/_{\sim}$. □

Hence, with the identification $\mathscr{U} = \mathscr{U}'$ if $\mathscr{U} \leq \mathscr{U}'$ and $\mathscr{U}' \leq \mathscr{U}$ we obtain a poset which turns to be a filtered poset. Indeed, if $\mathscr{U} = \{U_i\}_{i\in I}$ and $\mathscr{U}' = \{V_j\}_{j\in J}$ are open coverings of X, then $\mathscr{U} \cap \mathscr{U}' := \left\{U_i \cap U_j\right\}_{(i,j)\in I\times J}$ is a refinement of $\mathscr{U}$ and $\mathscr{U}'$ (take the projections for the λ's)[1]. We want to say that that we get that the "refinement relation" gives rise to a direct system of isomorphism classes of abelian groups. Thus, we need the following proposition.

Proposition 2.2.26
Let $\mathscr{U}$ *and* $\mathscr{U}'$ *be two equivalent open coverings of* X. *Then the induced map* $\check{H}^n(\mathscr{U},\mathscr{F}) \to \check{H}^n(\mathscr{U}',\mathscr{F})$ *is an isomorphism.*

Proof. We have the maps $\tau : \check{H}^n(\mathscr{U},\mathscr{F}) \to \check{H}^n(\mathscr{U}',\mathscr{F})$ and $\tau' : \check{H}^n(\mathscr{U}',\mathscr{F}) \to \check{H}^n(\mathscr{U},\mathscr{F})$. Since the map $\tau\tau'$ does not depend on the choice of the map from I to I, we can take the identity. Hence, $\tau\tau' = \mathrm{id}$. The symmetry implies $\tau'\tau = \mathrm{id}$, as required. □

Definition 2.2.27
Let $n \in \mathbb{N}_0$. *With the same notation as above, we define*

$$\check{H}^n(X,\mathscr{F}) = \varinjlim_{\mathscr{U}} \check{H}^n(\mathscr{U},\mathscr{F}),$$

where $\mathscr{U}$ *goes through the set of representatives of open coverings of* X.

Theorem 2.2.28
Let X *be a topological space and* $\mathscr{F}$ *a sheaf of abelian groups. Then* $\check{H}^1(X,\mathscr{F}) \cong H^1(X,\mathscr{F})$.

Proof. See [Har77], III.4 (Lemma 4.4, Theorem 4.5, and Exercise 4.4). □

[1] Since we use the definition of Čech cohomology with alternating cochains, we don't have to care about the order on $I \times J$

2.3 The Picard group and cohomology

The goal of this section is to show how we can realise the Picard group as a cohomology group. Namely we will show that $H^1\big(X, \mathcal{O}_X^*\big) \cong \operatorname{Pic} X$, for a scheme X. We follow exercise 5.3.2.7 of [Liu06].

Let X be a ringed space and let $\mathcal{L}$ be an invertible sheaf on X. Consider the open covering $\mathscr{U} = \{U_i\}_{i\in I}$ where $\mathcal{L}\big|_{U_i}$ is a free $\mathcal{O}_X\big|_{U_i}$-module of rank 1. Thus, we can find for each $i \in I$ an element $e_i \in \mathcal{L}(U_i)$ such that $\{e_i\}$ is a basis for $\mathcal{L}(U_i)$. We denote by $\Phi_i : \mathcal{L}\big|_{U_i} \longrightarrow \mathcal{O}_X\big|_{U_i}$ the isomorphism and use it to see that $e_i\big|_V$ is a basis for $\mathcal{L}(V)$, as an $\mathcal{O}_X(V)$-module, for all $V \subset U_i$. Therefore, there exists for each $i, j \in I$ an element $f_{ij} \in \mathcal{O}_X(U_{ij})$ such that $e_i\big|_{U_{ij}} = f_{ij} \cdot e_j\big|_{U_{ij}}$. Moreover, the unicity of the decomposition of the elements implies that $f_{ij} \in \mathcal{O}_X(U_{ij})^*$. We see that $f_{ii} = 1$ and the equalities $e_i\big|_{U_{ij}} = f_{ij} \cdot e_j\big|_{U_{ij}}$ and $e_j\big|_{U_{ij}} = f_{ji} \cdot e_i\big|_{U_{ij}}$ imply $e_i\big|_{U_{ij}} = f_{ij} f_{ji} \cdot e_j\big|_{U_{ij}}$ which means that $f_{ij} f_{ji} = 1$. Hence, we get an element $f \in C'^1(\mathscr{U}, \mathcal{O}_X^*)$.

We consider $(i, j, k) \in I^3$ and use the previous equality to get

$$e_i\big|_{U_{ijk}} = f_{ij}\big|_{U_{ijk}} \cdot e_j\big|_{U_{ijk}}, \quad e_j\big|_{U_{ijk}} = f_{jk}\big|_{U_{ijk}} \cdot e_k\big|_{U_{ijk}}, \quad e_i\big|_{U_{ijk}} = f_{ik}\big|_{U_{ijk}} \cdot e_k\big|_{U_{ijk}}$$

and so

$$e_j\big|_{U_{ijk}} = f_{jk}\big|_{U_{ijk}} \cdot f_{ik}\big|_{U_{ijk}}^{-1} \cdot f_{ij}\big|_{U_{ijk}} \cdot e_j\big|_{U_{ijk}}.$$

The unicity of the decomposition implies that $1 = f_{jk}\big|_{U_{ijk}} \cdot f_{ik}\big|_{U_{ijk}}^{-1} \cdot f_{ij}\big|_{U_{ijk}}$ which means that $f \in \ker d^1$. Hence, we can consider the image $\Phi(f)$ in $\check{H}^1\big(\mathscr{U}, \mathcal{O}_X^*\big)$. We want to check that this image in $\check{H}^1\big(\mathscr{U}, \mathcal{O}_X^*\big)$ uniquely determined by $\mathcal{L}$. Suppose $\{e'_i\}_{i\in I}$ is another collection of elements such that $\{e'_i\}$ is a basis for $\mathcal{L}(U_i)$. The elements e_i and e'_i must differ by an invertible element. Hence, there exists a collection $\{g_i\}$, with $g_i \in \mathcal{O}_X(U_i)^*$ such that $e'_i = g_i e_i$. The collection $\{e'_i\}$ gives rise to $\{f'_{ij}\}$ such that $e'_i\big|_{U_{ij}} = f'_{ij} \cdot e'_j\big|_{U_{ij}}$, as above. Using again the unicity of the decomposition we see that $f_{ij} = f'_{ij} \cdot g_j\big|_{U_{ij}} \cdot g_i\big|_{U_{ij}}^{-1}$. Hence, f and f' will have the same image in $\check{H}^1\big(\mathscr{U}, \mathcal{O}_X^*\big)$ and we denote by $\phi_{\mathscr{U}}(\mathcal{L})$ this image.

We denote by $\phi(\mathcal{L})$ the image of $\phi_{\mathscr{U}}(\mathcal{L})$ in $\check{H}^1(X, \mathcal{O}_X^*)$ under the canonical morphism $\check{H}^1(\mathscr{U}, \mathcal{O}_X^*) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*)$ (see Definition 2.2.27). We choose two trivializing coverings $\mathscr{U} = \{U_i\}_{i\in I}$ and $\mathscr{U}' = \{V_j\}_{j\in J}$ of X and get the collections of elements $\{e_i\}_{i\in I}$ and $\{e'_j\}_{j\in J}$ as above which give rise to the $\{f_{ij}\}_{(ij)\in I^2}$ and $\{f'_{kl}\}_{(k,l)\in J^2}$. We want to verify that the images of f and f' in $\check{H}^1\big(\mathscr{U} \cap \mathscr{U}', \mathcal{O}_X^*\big)$ are the same. This is equivalent to see that the elements

$$(\lambda f)_{(i,k),(j,l)} = f_{i,j}\big|_{U_{ij}\cap V_{kl}} \text{ and } (\lambda' f')_{(i,k),(j,l)} = f'_{k,l}\big|_{U_{ij}\cap V_{kl}}$$

differ by an element of $\operatorname{im} d^0$. Proceeding as in the verification that $\phi(\mathcal{L})$ does not depend on the choice of the $\{e_i\}$ (see above), one can check that this is the case.

Summary
So far, we associated to every invertible sheaf $\mathcal{L}$ on X an element $\phi(\mathcal{L})$ of $\check{H}^1(X, \mathcal{O}_X^*)$. Note that this association gives rise to a map $\phi : \operatorname{Pic}(X) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*)$. Indeed, if $\mathcal{L} \cong \mathcal{L}'$ we have the elements

$$\{e_i\}, \quad \{e'_i\}, \quad \{f_{ij}\}, \quad \{f'_{ij}\}.$$

If we denote by φ the isomorphism between $\mathcal{L}$ and $\mathcal{L}'$, we have $\varphi_{U_{ij}}(e_i)\big|_{U_{ij}} = f_{ij} \cdot \varphi_{U_{ij}}(e_j)\big|_{U_{ij}}$. Since $\phi(\mathcal{L}')$ does not depend on the choice of the elements $\{e_i'\}$, we can take $e_i' = \varphi_{U_i}(e_i)$. Thus, $\phi(\mathcal{L}) = \phi(\mathcal{L}')$.

Proposition 2.3.1
Let X be a scheme. Then, the map $\phi : \mathrm{Pic}(X) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*)$ is a homomorphism.

Proof. Let $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}(X)$ be two invertible sheaves. First, we can choose an open covering $\mathscr{U} = \{U_i\}_{i\in I}$ such that for every $i \in I$:

$$\mathcal{L}\big|_{U_i} \cong \mathcal{O}_X\big|_{U_i}, \quad \mathcal{L}'\big|_{U_i} \cong \mathcal{O}_X\big|_{U_i}, \quad U_i \text{ is affine.}$$

As above, we have elements

$$\{e_i\}, \quad \{e_i'\}, \quad \{f_{ij}\}, \quad \{f_{ij}'\}.$$

We know that $e_i \otimes e_i'$ will be a basis for $\mathcal{L}(U_i) \otimes \mathcal{L}'(U_i)$ over $\mathcal{O}_X(U_i)$. Furthermore, we have

$$(e_i \otimes e_i')\big|_{U_{ij}} = e_i\big|_{U_{ij}} \otimes e_i'\big|_{U_{ij}} = f_{ij}e_j\big|_{U_{ij}} \otimes f_{ij}'e_j'\big|_{U_{ij}} = f_{ij}f_{ij}' \cdot (e_j \otimes e_j')\big|_{U_{ij}}.$$

We have to find a basis for $(\mathcal{L} \otimes \mathcal{L}')(U_i)$ over $\mathcal{O}_X(U_i)$. The important fact is that we have the isomorphism $(\mathcal{L} \otimes \mathcal{L}')(U_i) = \mathcal{L}(U_i) \otimes \mathcal{L}'(U_i)$. Indeed, if R is such that $\mathcal{O}_X\big|_{U_i} \cong \operatorname{Spec} R$, we get

$$\begin{aligned}
\mathcal{L}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{L}'(U_i) &\cong \mathcal{O}_X\big|_{U_i}(U_i) \otimes_{\mathcal{O}_X|_{U_i}(U_i)} \mathcal{O}_X\big|_{U_i}(U_i) \\
&\cong R \otimes_R R \cong R \\
&\cong \Gamma\big(\tilde{R}, \operatorname{Spec} R\big) = \Gamma\Big(\widetilde{R \otimes_R R}, \operatorname{Spec} R\Big) \\
&\cong \Gamma\Big(\tilde{R} \otimes_{\tilde{R}} \tilde{R}, \operatorname{Spec} R\Big) \cong \Gamma\Big(\widetilde{R \otimes_R R}, \operatorname{Spec} R\Big) \\
&\cong \Big(\mathcal{L}\big|_{U_i} \otimes_{\mathcal{O}_X|_{U_i}} \mathcal{L}'\big|_{U_i}\Big)(U_i) \cong \Big(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'\Big)\big|_{U_i}(U_i) \\
&\cong (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}')(U_i).
\end{aligned}$$

Therefore, we can take $e_i \otimes e_i'$ as a basis for $(\mathcal{L} \otimes \mathcal{L}')(U_i)$ and the "corresponding f" is $f_{ij}f_{ij}'$. This shows that ϕ is indeed a homomorphism of groups. □

Proposition 2.3.2
Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{O}_X$. Then, $\phi(\mathcal{L}) = 1$ if and only if $\mathcal{L}$ is free of rank 1.

Proof. Suppose that $\phi(\mathcal{L}) = 1$. Since the image of $\mathcal{L}$ does not depend on the choice of the open covering $\mathscr{U}$, we can assume that $\overline{f} = 1$ in $\check{H}^1(\mathscr{U}, \mathcal{O}_X^*)$. Therefore, there exists $g \in C^0(\mathscr{U}, \mathcal{O}_X^*)$ such that $f = d^0 g$. This implies that $\big(g_i e_i\big)\big|_{U_{ij}} = \big(g_j e_j\big)\big|_{U_{ij}}$ for all $i, j \in I$. Hence, the elements $e_i f_i$ glue to an element $h \in \mathcal{L}(X)$ which can be chosen as a generator. This implies that $\mathcal{L}$ is free of rank 1.

Suppose that $\mathcal{L}$ is free. This implies that $\mathcal{L}(X)$ admits a basis of the form $\{e\}$ for some $f \in \mathcal{O}_X(X)$. We choose the open covering $\{X\}$ and get the element $f = 1$ which means that f has image 1 in $\check{H}^1(X, \mathcal{O}_X^*)$, as required. □

Theorem 2.3.3
Let X be a scheme. Then, we have $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$.

Proof. We have a homomorphism $\phi : \operatorname{Pic}(X) \longrightarrow \check{H}^1(X, \mathcal{O}_X^*)$ and the previous proposition implies that ϕ is injective. So all we need to do is establish surjectivity: let $\left[\mathscr{U}, \overline{f}\,\right] \in \check{H}^1(X, \mathcal{O}_X^*)$. Set $\mathcal{L}_i = \mathcal{O}_X\big|_{U_i}$ and use the multiplication by the f_{ij} to get isomorphisms $\mathcal{L}_i\big|_{U_{ij}} \xrightarrow[\cong]{\varphi_{ij}} \mathcal{L}_j\big|_{U_{ij}}$. Since f is an alternating cochain, we have $\varphi_{ii} = \mathrm{id}$. Since $f \in \ker d^1$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Therefore, we can glue the $\mathcal{L}_i$ to get an invertible sheaf $\mathcal{L}$. We take $e_i \in \mathcal{L}(U_i)$ as the image of 1 under the isomorphism $\psi_i : \mathcal{O}_X\big|_{U_i}(U_i) \longrightarrow \mathcal{L}(U_i)$. We have the following commutative diagramm:

$$\begin{tikzcd}
 & \mathcal{L}_i\big|_{U_{ij}} \arrow[r, "\varphi_{ij}"] \arrow[d, equal] & \mathcal{L}_i\big|_{U_{ij}} \arrow[d, equal] \arrow[dr, bend left, "{\psi_j}^{-1}"] & \\
 & \mathcal{O}_X\big|_{U_{ij}} & \mathcal{O}_X\big|_{U_{ij}} \arrow[dr, "{\psi_j}^{-1}"] & \\
\mathcal{L}\big|_{U_{ij}} \arrow[ur, bend left, "\psi_i"] \arrow[uur, bend left, "\psi_i"] \arrow[rrr, "\mathrm{id}"'] & & & \mathcal{L}\big|_{U_{ij}}
\end{tikzcd}$$

The fact that ${\psi_j}^{-1}\, \varphi_{ij}\, \psi_j = \mathrm{id}$ comes from the glueing. Then, looking at the images of $e_i\big|_{U_{ij}}$, we have $e_i\big|_{U_{ij}} = f_{ij} \cdot e_j\big|_{U_{ij}}$. Hence, $\phi(\mathcal{L}) = \left[\mathscr{U}, \overline{f}\,\right]$ and we have shown that ϕ is surjective. Thus, we have $\operatorname{Pic}(X) \cong \check{H}^1(X, \mathcal{O}_X^*)$. We finish using Theorem 2.2.28.

□

Appendix A

Appendix

A.1 Some results of algebra

Proposition A.1.1
Let R be a ring, S a multiplicative subset of R and $\{M_i\}_{i\in I}$ a family of R-modules. Then $S^{-1}\big(\bigoplus_i M_i\big) \cong \bigoplus_i \big(S^{-1}M_i\big)$.

Proof. It is easy to see that $S^{-1}\big(\bigoplus_i M_i\big)$ satisfy the universal property of the direct sum. □

A.1.1 Graded modules

Notation A.1.2
Let R be a graded ring. We write R_+ for the ideal $\bigoplus_{d\in\mathbb{Z}} R_d$.

Definition A.1.3 (Graded module)
A graded module *R-module M is a R-module M such that:*

- *$R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a graded ring;*
- *$M = \bigoplus_{n\in\mathbb{N}_0} M_n$ (each M_n is a subgroup of M);*
- *$R_n M_m \subset M_{n+m}$.*

Definition A.1.4 (Twisted graded module)
Let M be a graded R-module. For $n \in \mathbb{Z}$, we define the graded R-module $M(n)$ by $M(n)_d = M_{d+n}$.

Remark A.1.5
If R is a graded ring (thus a graded module over itself), then $R(n)$ is a graded module over R (and not $R(n)$). This is because of this operation that we take a decomposition of M over $\mathbb{Z}$ and not over $\mathbb{N}$.

Definition A.1.6 (Degree)
Let M be a graded R-module, T be a multiplicative subset of R and $\frac{m}{r} \in T^{-1}R$. The degree, *$\partial\frac{m}{r}$, of $\frac{m}{r}$ is $\partial m - \partial r$.*

Notation A.1.7
Let M be a graded R-module and $\mathfrak{p}$ be a prime ideal of R. Define the multiplicative T as the set of all homogeneous elements which are not in $\mathfrak{p}$. Then we denote by $M_{(\mathfrak{p})}$ the set of elements of $T^{-1}R$ which are of degree 0.

Definition A.1.8 (Tensor product of graded modules)
Let M and N be two graded R-modules. We can put a structure of graded module on $M \otimes_R N$ as follows:

$$\big(M \otimes_R N\big)_k = \Big\{ \sum_i m_i \otimes n_i : \partial m_i + \partial n_i = k, m_i \text{ and } n_i \text{ homogeneous} \Big\}.$$

Proposition A.1.9
Let M be a graded R-module and $n \in \mathbb{N}$, Then $M \otimes_R R(n) \cong M(n)$, as graded R-modules.

A.2 Projective schemes

In this section, R denote a graded ring.

Notation A.2.1
We write $\operatorname{Proj} R$ for the set of homogeneous prime ideals of R which do not contain all of R_+.

Notation A.2.2
For an homogeneous ideal I of R, we write $\mathcal{V}(I)$ for

$$\big\{\mathfrak{p} \in \operatorname{Proj} R : I \subset \mathfrak{p}\big\}.$$

Because $\mathcal{V}(IJ) = \mathcal{V}(I) \cup \mathcal{V}(J)$ and $\mathcal{V}\big(\sum_j I_j\big) = \bigcap_j \mathcal{V}(I_j)$, we have the following definition.

Definition A.2.3 (Zariski topology on $\operatorname{Proj} R$)
The topology on $\operatorname{Proj} R$ *which is obtained by taking sets of the form $\mathcal{V}(I)$ as closed sets is the* Zariski topology *on* $\operatorname{Proj} R$.

Let $\mathfrak{p} \in \operatorname{Proj} R$ and $T_\mathfrak{p}$ be the multiplicative subset of all homogeneous elements which are not in $\mathfrak{p}$.

Definition A.2.4 (Degree)
Let $\frac{a}{q} \in T_\mathfrak{p}^{-1} R$. The degree, *$\partial \frac{a}{q}$, of $\frac{a}{q}$ is $\partial a - \partial q$.*

Notation A.2.5
We denote by $R_{(\mathfrak{p})}$ the set of elements of $T_\mathfrak{p}^{-1} R$ which are of degree 0.

Notation A.2.6
Let f be an homogeneous element of R_+. We denote by $R_{(f)}$ the set of elements of R_f which are of degree 0.

Remark A.2.7
Consider $n \in \mathbb{N}$, $R(n)$ and $f \in R_+$. Then

$$R(n)_{(f)} = \left\{ \frac{r}{f^m} \in R_f : \partial\left(\frac{r}{f^m}\right) = n \right\}.$$

Now, we are ready to define a sheaf of rings on $\operatorname{Proj} S$. Consider an open set U of $\operatorname{Proj} R$ and set $\mathcal{O}(U)$ as the set of functions $s : U \longrightarrow \coprod_{\mathfrak{p} \in U} R_{(\mathfrak{p})}$ which satisfy the following properties:

(i) $s(\mathfrak{q}) \in R_{(\mathfrak{p})}$, for all $\mathfrak{q} \in U$.

(ii) For each $\mathfrak{p} \in U$, there exists an open neighbourhood V of $\mathfrak{p}$ contained in U and homogeneous elements $r, p \in R$ of the same degree such that $p \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$ and $s(\mathfrak{q}) = \frac{r}{p}$.

Notation A.2.8
Let $f \in R_+$ be an homogeneous element. We set $D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} R : f \notin \mathfrak{p}\}$.

Definition A.2.9
For any graded ring R, we define $(\operatorname{Proj} R, \mathcal{O})$ as above.

With these choices, we have analogous properties as the affine case:

Proposition A.2.10
Let R be a graded ring. We have the following:

(i) For any $\mathfrak{p} \in \operatorname{Proj} R$, we have $\mathcal{O}_{\mathfrak{p}} \cong R_{(\mathfrak{p})}$, as local rings.

(ii) For any homogeneous element $f \in R_+$, we have $\mathcal{O}\big(D_+(f)\big) \cong R_{(f)}$.

(iii) As $f \in R_+$ is going through all homogeneous element of R_+, the sets $D_+(f)$ cover $\operatorname{Proj} R$.

(iv) $\big(D_+(f), \mathcal{O}\big|_{D_+(f)}\big) \cong \operatorname{Spec} R_{(f)}$, as locally ringed space.

(v) $\operatorname{Proj} R$ is a scheme.

Proof. See II.2 of [Har77]. □

Table of notations

$\mathfrak{Ab}(X)$	Category of sheaves on a topological space X
$\mathrm{CaCl}(X)$	Group of Cartier divisor of a scheme X modulo principal divisor
$\mathrm{CaDiv}(X)$	Group of Cartier divisor of a scheme X
$\mathrm{Cl}(X)$	Divisor class group of a scheme X
$\mathrm{Div}(X)$	Group of Weil divisor of a scheme
$\mathscr{M}od(\mathcal{O}_X)$	Category of $\mathcal{O}_X$-modules
$\mathbb{N}$	Set of postive integers $\{1, 2, \ldots\}$
$\mathbb{N}_0$	Set of non-negative integers $\{0, 1, 2, \ldots\}$
$\mathrm{Pic}(X)$	Picard group of X

Index

Bibliography

[Gug10] Rafael Guglielmetti, *Introduction à la géométrie algébrique*, `http://www.allpotes.ch/raf/epfl/groupes_profinis.pdf`, 2010.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer Verlag, 1977.

[Liu06] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford University Press, USA, 2006.

[Mat70] H. Matsumura, *Commutative algebra*, vol. 120, WA Benjamin, 1970.

[Mur06] Daniel Murfet, *Divisors*, `http://therisingsea.org/notes/Section2.6-Divisors.pdf`, 2006.

[Rot08] J.J. Rotman, *An introduction to homological algebra*, Springer Verlag, 2008.

[Ser55] J.P. Serre, *Faisceaux algébriques cohérents*, The Annals of Mathematics **61** (1955), no. 2, 197–278.